

# How Inductive Bias in Machine Learning Aligns with Optimality in Economic Dynamics

---

Mahdi Ebrahimi Kahou<sup>1</sup> James Yu<sup>2</sup> Jesse Perla<sup>2</sup> Geoff Pleiss<sup>3,4</sup>

July 26, 2024

<sup>1</sup>Bowdoin College, Econ Dept

<sup>2</sup>University of British Columbia, Vancouver School of Economics

<sup>3</sup>University of British Columbia, Stats Dept

<sup>4</sup>Vector Institute

*In the long run, we are all dead—J.M. Keynes, A Tract on Monetary Reform (1923)*

- Numerical solutions to dynamical systems are central to many quantitative fields in economics.
- Dynamical systems in economics are **boundary value** problems:
  1. The boundary is at **infinity**.
  2. The values at the boundary are potentially **unknown**.
- Resulting from **forward looking** behavior of agents.
- Examples include the transversality and the no-bubble condition.
- Without them, the problems are **ill-posed** and have infinitely many solutions:
  - These forward-looking boundary conditions are a key limitation on increasing dimensionality.

## 1. Inductive bias alignment:

- The minimum-norm implicit bias of modern ML models automatically satisfies economic boundary conditions at infinity.

## 2. Learning the right set of steady-states:

- Deep neural networks and kernel machines learn the boundary values, thereby extrapolating very accurately.

## 3. Robustness and speed:

- Competitive in speed and more stable than traditional methods.

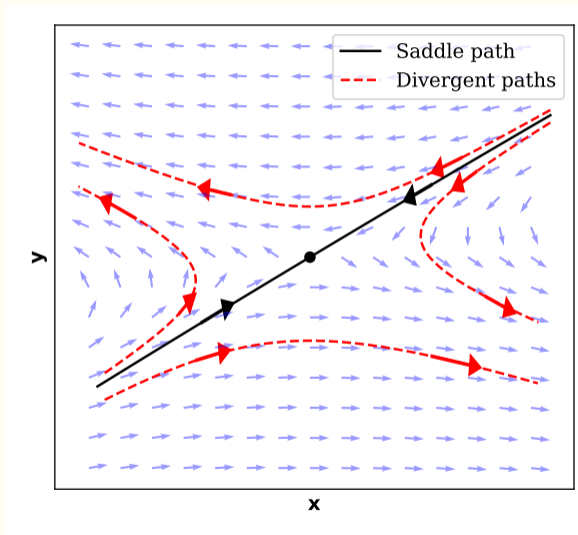
## 4. Consistency of ML estimates.

- **Minimum-norm implicit bias:**

- Over-parameterized models (e.g., large neural networks) have more parameters than data points and potentially interpolate the data.
- They are biased towards interpolating functions with smallest norm.

- **Violation of economic boundary conditions:**

- Sub-optimal solutions diverge (explode) over time.
- They have large or explosive norms.
- This is due to the **saddle-path** nature of econ problems.



# The Problem

---

# The class of problems

A differential-algebraic system of equations, coming from an economic optimization problem:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{y}(t)) \quad (1)$$

$$\dot{\mathbf{y}}(t) = \mathbf{G}(\mathbf{x}(t), \mathbf{y}(t)) \quad (2)$$

$$\mathbf{0} = \mathbf{H}(\mathbf{x}(t), \mathbf{y}(t)) \quad (3)$$

$\mathbf{x} \in \mathbb{R}^{N_x}$ : state variables,  $\mathbf{y} \in \mathbb{R}^{N_y}$ : jump variables. Initial value  $\mathbf{x}(0) = \mathbf{x}_0$  and boundary conditions (at infinity)

$$\mathbf{0} = \lim_{t \rightarrow \infty} \mathbf{B}(t, \mathbf{x}(t), \mathbf{y}(t)) \quad (4)$$

**Goal:** finding an approximation for  $\mathbf{x}(t)$  and  $\mathbf{y}(t)$ .

**What is the problem?**

- $\mathbf{y}_0$  is unknown.
- The optimal solutions is a **saddle-path**: unstable nature

## Method

---

# Method

- Pick a set of points  $\mathcal{D} \equiv \{t_1, \dots, t_N\}$  for some fixed interval  $[0, T]$
- Large machine learning models to learn  $\hat{\mathbf{x}}(t)$  and  $\hat{\mathbf{y}}(t)$

$$\min_{\hat{\mathbf{x}}, \hat{\mathbf{y}}} \sum_{t_i \in \mathcal{D}} \left[ \underbrace{\eta_1 \left\| \hat{\mathbf{x}}(t_i) - \mathbf{F}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)(t_i)) \right\|_2^2}_{\text{Residuals}^2: \text{ state variables}} + \underbrace{\eta_2 \left\| \hat{\mathbf{y}}(t_i) - \mathbf{G}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)) \right\|_2^2}_{\text{Residuals}^2: \text{ jump variables}} \right. \\ \left. + \eta_3 \underbrace{\left\| \mathbf{H}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)) \right\|_2^2}_{\text{Residuals}^2: \text{ algebraic constraint}} \right] + \eta_4 \underbrace{\left\| \hat{\mathbf{x}}(0) - \mathbf{x}_0 \right\|_2^2}_{\text{Residuals}^2: \text{ initial conditions}} .$$

- This optimization **ignores** the boundary conditions.
- The implicit bias automatically satisfy the boundary conditions.
- Recent works suggest the implicit bias is toward smallest Sobolev semi-norms.

# Ridgeless kernel regression

$$\hat{\mathbf{x}}(t) = \sum_{j=1}^N \alpha_j^x K(t, t_j), \quad \hat{\mathbf{y}}(t) = \sum_{j=1}^N \alpha_j^y K(t, t_j)$$

$$\hat{\mathbf{x}}(t) = \mathbf{x}_0 + \int_0^t \hat{\mathbf{x}}(\tau) d\tau, \quad \hat{\mathbf{y}}(t) = \hat{\mathbf{y}}_0 + \int_0^t \hat{\mathbf{y}}(\tau) d\tau$$

- $\mathbf{x}_0$  is given.
- $\hat{\mathbf{y}}_0$ ,  $\alpha_j^x$ , and  $\alpha_j^y$  are learnable parameters.
- $K(\cdot, \cdot)$ : Matérn Kernel, with smoothness parameter  $\nu$  and length scale  $\ell$ .

# Ridgeless kernel regression: minimum Sobolev seminorm solutions

We also solve the ridgeless kernel regression

$$\lim_{\lambda \rightarrow 0} \min_{\hat{\mathbf{x}}, \hat{\mathbf{y}}} \sum_{t_i \in \mathcal{D}} \left[ \eta_1 \left\| \hat{\mathbf{x}}(t_i) - \mathbf{F}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)(t_i)) \right\|_2^2 + \eta_2 \left\| \hat{\mathbf{y}}(t_i) - \mathbf{G}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)) \right\|_2^2 \right. \\ \left. + \eta_3 \left\| \mathbf{H}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)) \right\|_2^2 \right] + \eta_4 \left\| \hat{\mathbf{x}}(0) - \hat{\mathbf{x}}_0 \right\|_2^2 + \lambda \underbrace{\left[ \sum_{m=1}^{N_x} \left\| \hat{\mathbf{x}}^{(m)} \right\|_{\mathcal{H}}^2 + \sum_{m=1}^{N_y} \left\| \hat{\mathbf{y}}^{(m)} \right\|_{\mathcal{H}}^2 \right]}_{\text{The Sobolev semi-norm}}$$

- Targeting Sobolev semi-norm.
- This choice is very natural: it solves the instability issues of the classical algorithm.

# Applications

---

$$\dot{\mathbf{x}}(t) = c + g\mathbf{x}(t) \quad (5)$$

$$\dot{\mathbf{y}}(t) = r\mathbf{y}(t) - \mathbf{x}(t) \quad (6)$$

$$0 = \lim_{t \rightarrow \infty} e^{-rt} \mathbf{y}(t) \quad (7)$$

- $\mathbf{x}(t) \in \mathbb{R}$ : dividends,  $\mathbf{y}(t) \in \mathbb{R}$ : prices, and  $\mathbf{x}_0$  given.
- Equation (5): how the dividends evolve in time.
- Equation (6): how the prices evolve in time.
- Equation (7): “no-bubble” condition, the boundary condition at infinity.

# Why do we need the boundary condition?

$$\dot{\mathbf{x}}(t) = c + g\mathbf{x}(t)$$

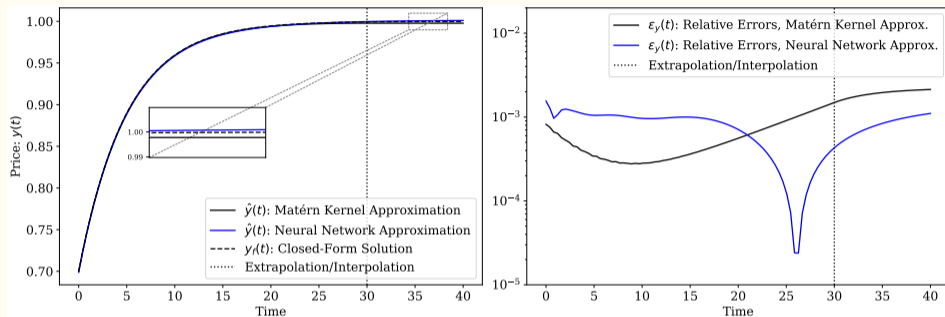
$$\dot{\mathbf{y}}(t) = r\mathbf{y}(t) - \mathbf{x}(t)$$

- The solutions:

$$\mathbf{y}(t) = \mathbf{y}_f(t) + \zeta e^{rt}$$

- $\mathbf{y}_f(t) = \int_0^\infty e^{-r\tau} \mathbf{x}(t+s) ds$ : price based on the fundamentals.
- $\zeta e^{rt}$ : explosive bubble terms, it has to be **ruled out** by the boundary condition.
- Triangle inequality:  $\|\mathbf{y}_f\| < \|\mathbf{y}\|$ .
- The price based on the fundamentals has the **lowest norm**.

# Results



$$\mathcal{D} = \{0, 1, \dots, 30\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Very accurate approximations, both in the short- and medium-run.
- Learns the steady-state.

# Neoclassical growth model: the agent's problem

$$\begin{aligned} \max_{\mathbf{y}(t)} \int_0^{\infty} e^{-rt} \ln(\mathbf{y}(t)) dt \\ \text{s.t. } \dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t) \end{aligned}$$

for a given  $\mathbf{x}_0$ .

- $\mathbf{x}(t) \in \mathbb{R}$ : capital,  $\mathbf{y}(t) \in \mathbb{R}$ : consumption, and a concave production function  $f(x) = x^a$ .

Constructing the Hamiltonian ...

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t) \quad (8)$$

$$\dot{\mathbf{y}}(t) = \mathbf{y}(t) [f'(\mathbf{x}(t)) - \delta - r] \quad (9)$$

$$0 = \lim_{t \rightarrow \infty} e^{-rt} \frac{\mathbf{x}(t)}{\mathbf{y}(t)} \quad (10)$$

- Equation (10) : transversality condition (TVC)

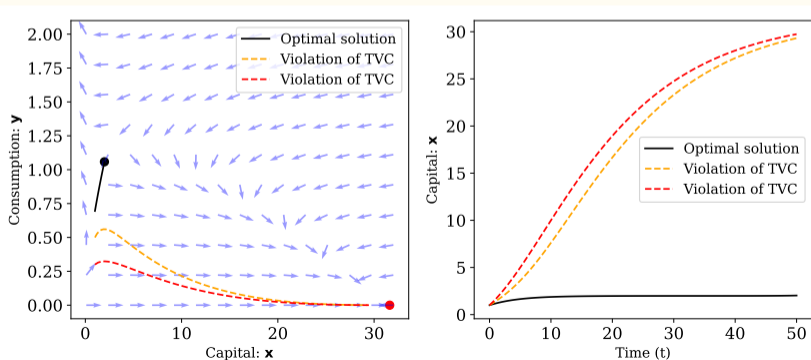
# Why do we need the boundary condition?

Ignoring the transversality condition:

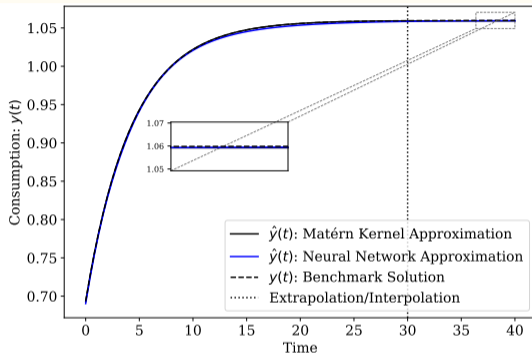
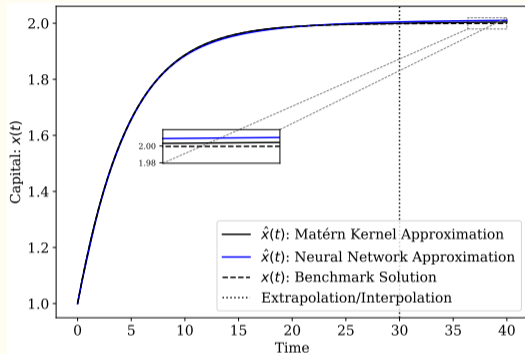
$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{y}(t)[f'(\mathbf{x}(t)) - \delta - r]$$

$$\mathbf{x}(0) = \mathbf{x}_0 \text{ given.}$$



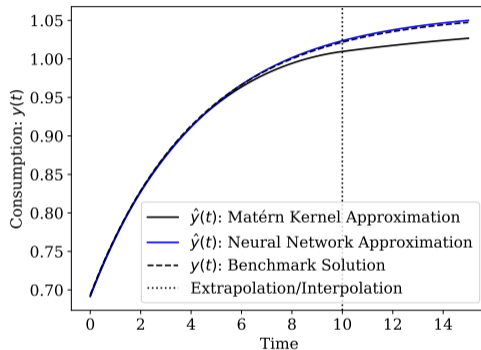
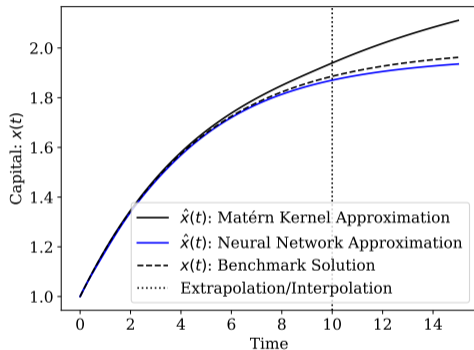
# Results



$$\mathcal{D} = \{0, 1, \dots, 30\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Very accurate approximations, both in the short- and medium-run.
- Learns the **right steady-state**. » Relative errors

## Short-run planning: “In the long run, we are all dead”



$$\mathcal{D} = \{0, 1, \dots, 10\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Provides a very accurate approximation in the short-run.

## Extensions

---

# Neoclassical Growth Model: Non-Concave Production Function

- So far we have had a **unique** saddle-path converging to a unique **saddle** steady state.
- What if we have **two** saddle steady states, very close to each other (equilibrium multiplicity)?
- Neoclassical growth model with a non-concave production function (threshold externalities):

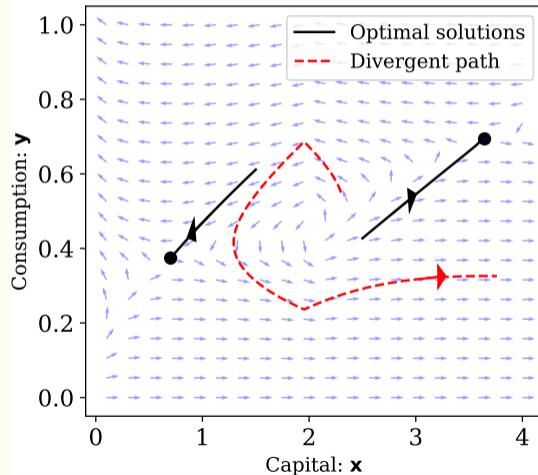
$$f(x) = A \max\{x^a, b_1 x^a - b_2\}$$

# Non-concave production function: vector field

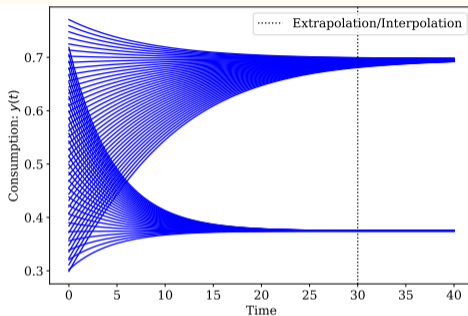
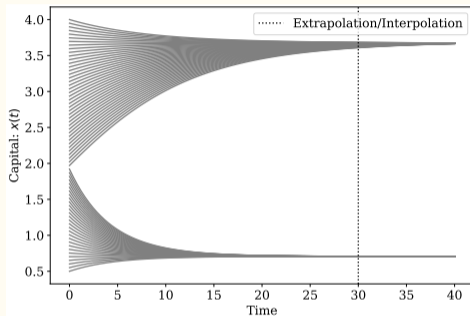
$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{y}(t)[f'(\mathbf{x}(t)) - \delta - r]$$

$$\mathbf{x}(0) = \mathbf{x}_0 \text{ given.}$$



# Results



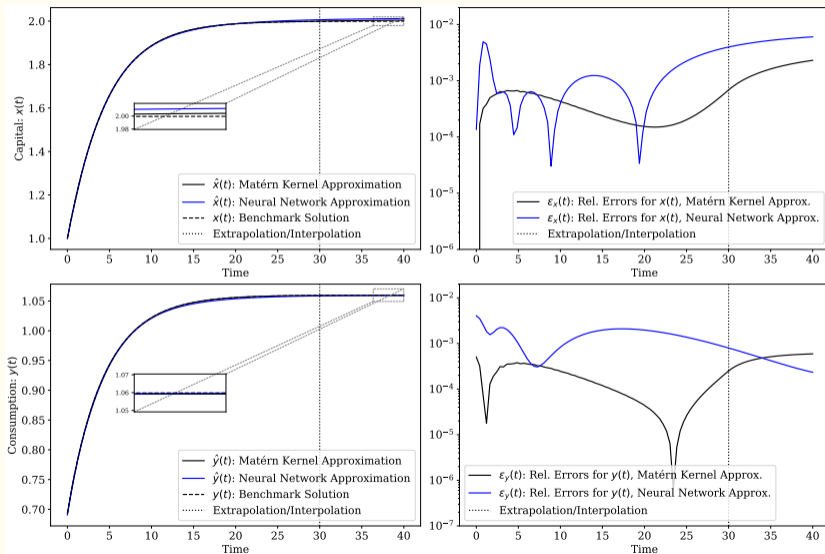
- The approximate solutions approach the right steady states.
- The transversality conditions are satisfied without being directly imposed.
- The steady states are learned. [▶ Full DAE](#)

- Long-run (**global**) conditions can be replaced with appropriate regularization (**local**) to achieve the optimal solutions.
- The minimum-norm implicit bias of large ML models aligns with optimality in economic dynamic models.
- Both kernel and neural network approximations accurately learn the right steady state(s).
- Proceeding with **caution**: can regularization be thought of as an equilibrium selection device?

# Appendix

---

# Neoclassical growth: relative errors



$$\dot{\mathbf{x}}_k(t) = \mathbf{y}_k(t) - \delta_k \mathbf{x}_k(t),$$

$$\dot{\mathbf{x}}_h(t) = \mathbf{y}_h(t) - \delta_h \mathbf{x}_h(t)$$

$$\dot{\mathbf{y}}_c(t) = \mathbf{y}_c(t) [f_1(\mathbf{x}_k(t), \mathbf{x}_h(t)) - \delta_k - r],$$

$$0 = f(\mathbf{x}_k(t), \mathbf{x}_h(t)) - \mathbf{y}_c(t) - \mathbf{y}_k(t) - \mathbf{y}_h(t),$$

$$0 = f_2(\mathbf{x}_k(t), \mathbf{x}_h(t)) - f_1(\mathbf{x}_k(t), \mathbf{x}_h(t)) + \delta_k - \delta_h.$$

$$0 = \lim_{t \rightarrow \infty} e^{-rt} \frac{\mathbf{x}_k(t)}{\mathbf{y}_c(t)}, \quad 0 = \lim_{t \rightarrow \infty} e^{-rt} \frac{\mathbf{x}_h(t)}{\mathbf{y}_c(t)}.$$

- $\mathbf{x}_k$ : physical capital,  $\mathbf{x}_h$ : human capital,  $\mathbf{y}_c$ : consumption,  $\mathbf{y}_k$ : investment in physical capital,  $\mathbf{y}_h$ : investment in human capital
- $f(x_k, x_h) = x_k^{a_k} x_h^{a_h}$

# Results

