Solving Models of Economic Dynamics with Ridgeless Kernel Regressions

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Motivation

- Numerical solutions to dynamical systems are central to many quantitative fields in economics.
- Dynamical systems in economics are **boundary value** problems:
  1. The boundary is at **infinity**.
  2. The values at the boundary are potentially **unknown**.
- Resulting from **forward looking** behavior of agents.
- Examples include the transversality and the no-bubble condition.
- Without them, the problems are **ill-posed** and have infinitely many solutions:
  - These forward-looking boundary conditions are a key limitation on increasing dimensionality.

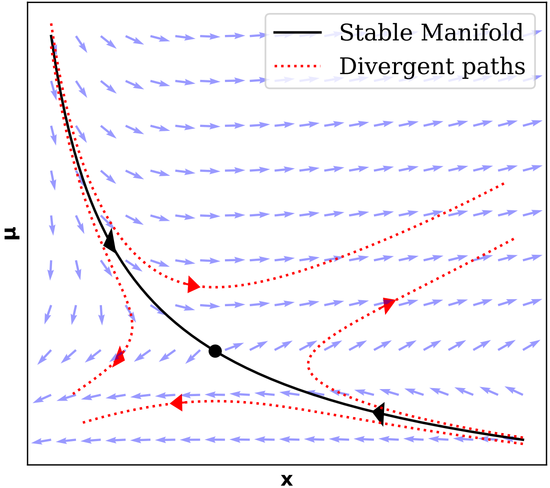
Contribution

Using kernel methods to solve a class of infinite-horizon, deterministic, continuous-time models

1. **Minimum-norm alignment**:
   - The minimum-norm kernel method aligns with asymptotic boundary conditions.
2. **Learning the correct set of steady-states**:
   - Kernel machines learn the boundary values, thereby extrapolating outside the training data.
3. **Robustness and speed**:
   - Competitive in speed and more stable than traditional methods.
4. **Consistency of the kernel estimates**.

Intuition

- **Violation of the boundary conditions**:
  - Sub-optimal solutions explode over time.
  - They have large derivatives.
  - This behavior is due to the **saddle-path** nature of the problem.
- **Minimum-norm solution** :
  - Penalizing large derivatives rules out explosive paths.
  - The remaining solution is the optimal solution.



The Problem

The class of problems

A differential-algebraic system of equations, coming from an optimization problem:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu}(t), \mathbf{y}(t)) \tag{1}$$

$$\dot{\boldsymbol{\mu}}(t) = r\boldsymbol{\mu}(t) - \boldsymbol{\mu}(t) \odot \mathbf{G}(\mathbf{x}(t), \boldsymbol{\mu}(t), \mathbf{y}(t)) \tag{2}$$

$$\mathbf{0} = \mathbf{H}(\mathbf{x}(t), \boldsymbol{\mu}(t), \mathbf{y}(t)), \tag{3}$$

boundary conditions (at infinity)

$$\mathbf{0} = \lim_{t \to \infty} e^{-rt}\mathbf{x}(t) \odot \boldsymbol{\mu}(t), \tag{4}$$

initial value $\mathbf{x}(0) = \mathbf{x}_0$.

- $\mathbf{x} \in \mathbb{R}^M$: state variables.
- $\boldsymbol{\mu} \in \mathbb{R}^M$: co-state variables.
- $\mathbf{y} \in \mathbb{R}^P$: jump variables.

Challenges

Goal: Find an approximation to $\mathbf{x}(t)$, $\mu(t)$, and $\mathbf{y}(t)$.

What is the problem?

- Initial conditions $\mathbf{y}_0$ and μ_0 are unknown.
- The optimal solution follows a **saddle path**.
    - If T is small, solutions are inaccurate due to premature enforcement of the steady state.
    - If T is large, the algorithms become increasingly numerically unstable.

Example: Neoclassical Growth Model

$$\dot{x}(t) = f(x(t)) - \delta x(t) - y(t) := F(x(t), \mu(t), y(t))$$

$$\dot{\mu}(t) = r\mu(t) - \mu(t)\,\underbrace{\left(f'(x(t)) - \delta\right)}_{:=G(x(t),\mu(t),y(t))}$$

$$0 = \mu(t)y(t) - 1 := H(x(t), \mu(t), y(t))$$

$$x(0) = x_0, \qquad \lim_{t\to\infty} e^{-rt}\mu(t)x(t) = 0$$

capital $x(t)$, consumption $y(t)$, utility $\log(y)$, present-value co-state variable $\mu(t)$, discount rate $r > 0$, depreciation rate $0 < \delta < 1$, and production function $f(x)$.

Method

Method: approximation

- Pick a set of points $\mathcal{D} \equiv \{t_1, \cdots, t_N\}$ for some fixed interval $[0, T]$

$$\hat{\dot{\mathbf{x}}}(t) = \sum_{j=1}^{N} \boldsymbol{\alpha}_j^x \, k(t, t_j), \qquad \hat{\dot{\boldsymbol{\mu}}}(t) = \sum_{j=1}^{N} \boldsymbol{\alpha}_j^\mu \, k(t, t_j), \qquad \hat{\dot{\mathbf{y}}}(t) = \sum_{j=1}^{N} \boldsymbol{\alpha}_j^y \, k(t, t_j),$$

$$\hat{\mathbf{x}}(t) = \mathbf{x}_0 + \int_0^t \hat{\dot{\mathbf{x}}}(\tau)\, d\tau, \qquad \hat{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}_0 + \int_0^t \hat{\dot{\boldsymbol{\mu}}}(\tau)\, d\tau, \qquad \hat{\mathbf{y}}(t) = \mathbf{y}_0 + \int_0^t \hat{\dot{\mathbf{y}}}(\tau)\, d\tau.$$

- $\boldsymbol{\alpha}_j^x$, $\boldsymbol{\alpha}_j^\mu$, $\boldsymbol{\alpha}_j^y$, $\boldsymbol{\mu}_0$, and $\mathbf{y}_0$ are parameters to be found.
- $k(t, t_j)$ is a kernel that measures how close (or similar) t is to t_j.
- We use a Matérn kernel with smoothness ν and length ℓ.

Matérn kernel

Method: Ridgeless kernel regression

We solve

$$\begin{aligned}
\min_{\hat{\mathbf{x}},\hat{\boldsymbol{\mu}},\hat{\mathbf{y}}} & \left(\sum_{m=1}^{M} \|\dot{\hat{\mathbf{x}}}^{(m)}\|_{\mathcal{H}}^2 + \sum_{m=1}^{M} \|\dot{\hat{\boldsymbol{\mu}}}^{(m)}\|_{\mathcal{H}}^2 \right) \\
\text{s.t. } & \dot{\hat{\mathbf{x}}}(t_i) = \mathbf{F}(\hat{\mathbf{x}}(t_i), \hat{\boldsymbol{\mu}}(t_i), \hat{\mathbf{y}}(t_i)), \quad \text{for all } t_i \in \mathcal{D} \\
& \dot{\hat{\boldsymbol{\mu}}}(t_i) = r\hat{\boldsymbol{\mu}}(t_i) - \hat{\boldsymbol{\mu}}(t_i) \odot \mathbf{G}(\hat{\mathbf{x}}(t_i), \hat{\boldsymbol{\mu}}(t_i), \hat{\mathbf{y}}(t_i)), \quad \text{for all } t_i \in \mathcal{D} \\
& \mathbf{0} = \mathbf{H}(\hat{\mathbf{x}}(t_i), \hat{\boldsymbol{\mu}}(t_i), \hat{\mathbf{y}}(t_i)), \quad \text{for all } t_i \in \mathcal{D}.
\end{aligned}$$

- $\|\dot{\hat{\mathbf{x}}}^{(m)}\|_{\mathcal{H}}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i^{x^{(m)}} \alpha_j^{x^{(m)}} k(t_i, t_j)$ and $\|\dot{\hat{\boldsymbol{\mu}}}^{(m)}\|_{\mathcal{H}}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i^{\mu^{(m)}} \alpha_j^{\mu^{(m)}} k(t_i, t_j)$
- TVC is **not imposed**.
- The minimization term (objective) is used to control the smoothness of the approximating functions.
- For Matérn kernels, it also controls the smoothness of derivatives.

Applications

Neoclassical growth model

$$\max_{y(t)} \int_0^\infty e^{-rt} \ln(y(t))dt$$
$$\text{s.t.} \quad \dot{x}(t) = f(x(t)) - y(t) - \delta x(t)$$

for a given x_0.

- $x(t) \in \mathbb{R}$: capital, $y(t) \in \mathbb{R}$: consumption, and a concave production function $f(x) = x^a$.

Constructing the Hamiltonian ...

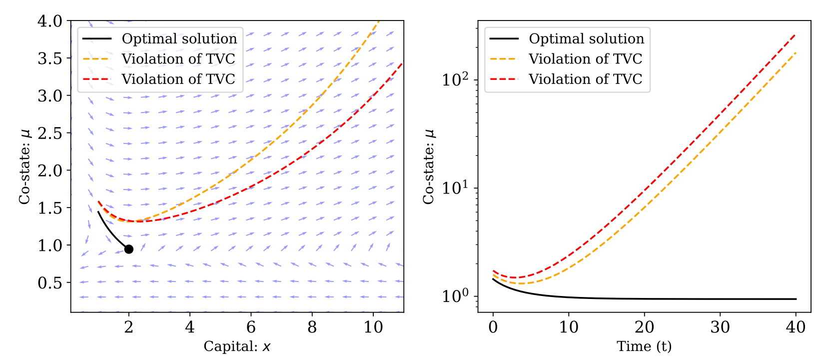
$$\begin{aligned} \dot{x}(t) &= f(x(t)) - \delta x(t) - y(t), \\ \dot{\mu}(t) &= r\mu(t) - \mu(t)\big(f'(x(t)) - \delta\big), \\ 0 &= \mu(t)y(t) - 1, \\ x(0) &= x_0, \qquad \lim_{t\to\infty} e^{-rt}\mu(t)x(t) = 0. \end{aligned}$$

- Last Equation : transversality condition (TVC)

Why do we need the boundary condition?

Ignoring the transversality condition:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) - \delta x(t) - y(t), \\
\dot{\mu}(t) &= r\mu(t) - \mu(t)\big(f'(x(t)) - \delta\big), \\
0 &= \mu(t)y(t) - 1, \\
x(0) &= x_0.
\end{aligned}
$$



Neoclassical Growth Model: algorithm

$$
\begin{aligned}
\min_{\hat{x},\hat{\mu},\hat{y}} \; & \left(\|\dot{\hat{x}}\|^2_{\mathcal{H}} + \|\dot{\hat{\mu}}\|^2_{\mathcal{H}} \right) \\
\text{s.t. } & \dot{\hat{x}}(t_i) = f(\hat{x}(t_i)) - \delta\hat{x}(t_i) - \hat{y}(t_i), \quad \text{for all } t_i \in \mathcal{D} \\
& \dot{\hat{\mu}}(t_i) = r\hat{\mu}(t_i) - \hat{\mu}(t_i)\big(f'(\hat{x}(t_i) - \delta)\big), \quad \text{for all } t_i \in \mathcal{D} \\
& 0 = \hat{\mu}(t_i)\hat{y}(t_i) - 1, \quad \text{for all } t_i \in \mathcal{D}.
\end{aligned}
$$

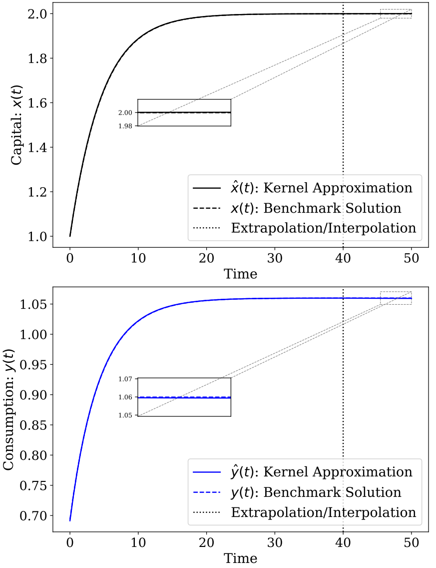
- $\|\dot{\hat{x}}\|^2_{\mathcal{H}} = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i^x \, \alpha_j^x \, k(t_i, t_j)$ and $\|\dot{\hat{\mu}}\|^2_{\mathcal{H}} = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i^\mu \, \alpha_j^\mu \, k(t_i, t_j)$
- TVC is **not imposed**.

Neoclassical growth model: results

- $\mathcal{D} = \{0, 1, \cdots, 40\}$. ⏩ sparse grids
- $f(x) = x^{\frac{1}{3}}$, $\delta = \frac{1}{3}$, and $r = 0.11$.
- The explosive solutions are ruled out without directly imposing the boundary condition.

  Conclusion

  - For short- and medium-run accuracy, we do not need to know the steady state (global condition).
  - It suffices that sub-optimal paths diverge from the optimal path (local condition).
- Learns the **correct steady state**. ⏩ Relative errors



Neoclassical growth model: learning the steady state

Why does it learn the **correct steady state**?

- A straight line is the "smoothest" solution: it has zero derivatives.
- The kernels we use are zero-reverting:

$$\lim_{t\to\infty} k(t, t_j) = 0.$$

- We approximate the derivatives using kernels:

$$\lim_{t\to\infty} \hat{\dot{x}}(t) = \hat{\dot{\mu}}(t) = \hat{\dot{y}}(t) = 0.$$

- This behavior is mainly driven by choosing a large t_N (e.g., $t_N = 40$) in $\mathcal{D}$.
- **Question**: How accurate are the short-run dynamics when t_N is small?

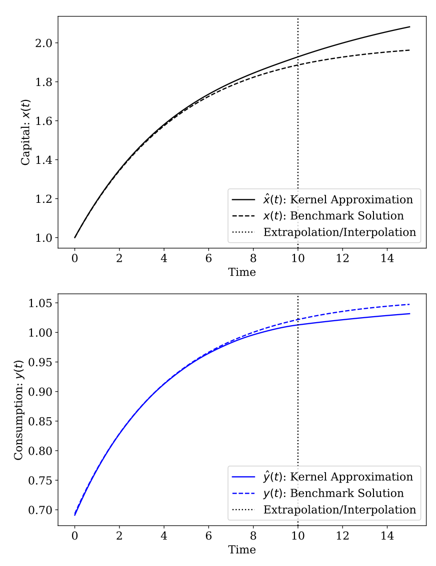
Neoclassical growth model: short-run results

- $\mathcal{D} = \{0, 1, \cdots, 10\}$.
- $f(x) = x^{\frac{1}{3}}$, $\delta = \frac{1}{3}$, and $r = 0.11$.
- Very accurate short-run dynamics.

Conclusion

- For short-run accuracy, we do not need to use large T (global condition).
- It suffices that sub-optimal paths diverge fast enough from the optimal path (local condition).

Relative errors



Neoclassical Growth Model: Non-Concave Production Function

- So far we have had a **unique** saddle-path converging to a unique **saddle** steady state.
- What if we have **two** saddle steady states, very close to each other (steady-state multiplicity)?
- Neoclassical growth model with a non-concave production function (threshold externalities):

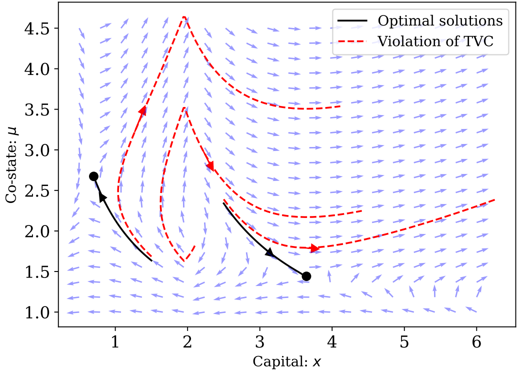
$$f(x) = A \max\{x^a, b_1 x^a - b_2\}$$

- The production function has a kink.

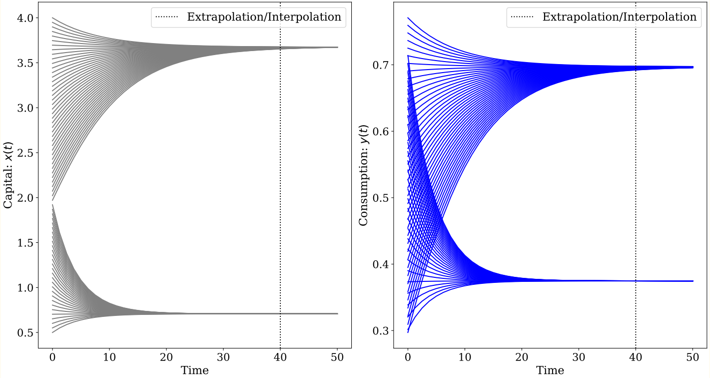
Non-concave production function: vector field

$$\dot{x}(t) = f(x(t)) - \delta x(t) - y(t),$$
$$\dot{\mu}(t) = r\mu(t) - \mu(t)\big(f'(x(t)) - \delta\big),$$
$$0 = \mu(t)y(t) - 1,$$
$$x(0) = x_0.$$

- The model is identical to the concave case, except that $f(\cdot)$ is now non-concave.
- This problem is very challenging for traditional methods such as shooting.



Results



- The approximate solutions approach the right steady states.
- The transversality conditions are satisfied without being directly imposed.
- The steady states are learned.

Extensions

Linear asset pricing

Linear asset pricing model

$$\dot{x}(t) = c + gx(t)$$
$$\dot{\mu}(t) = r\mu(t) - x(t) := r\mu(t) - \mu(t)\frac{x(t)}{\mu(t)}$$
$$0 = \lim_{t\to\infty} e^{-rt}\mu(t)x(t).$$

- $x(t) \in \mathbb{R}$: flow payoffs from a claim to an asset.
- $\mu(t) \in \mathbb{R}$ be the price of a claim to that asset.
- x_0 given.

Why do we need the boundary condition?

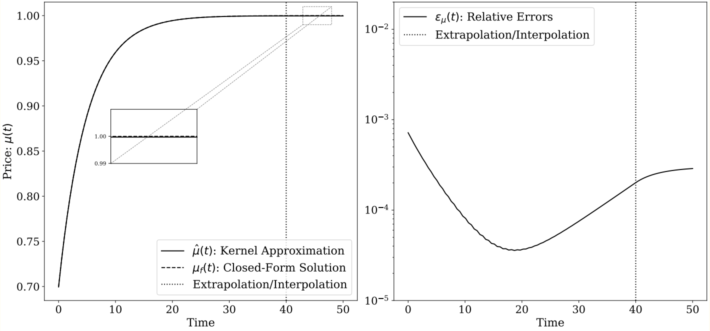
$$\dot{x}(t) = c + gx(t)$$
$$\dot{\mu}(t) = r\mu(t) - x(t)$$

- The solutions:

$$\mu(t) = \mu_f(t) + \zeta e^{rt}$$

- $\mu_f(t) = \int_0^\infty e^{-r\tau} x(t+\tau) d\tau$: price based on the fundamentals.
- ζe^{rt}: explosive bubble terms, it has to be **ruled out** by the boundary condition.
- The price based on the fundamentals is the **"smoothest"**.

Results



$$\mathcal{D} = \{0, 1, \cdots, 30\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Very accurate approximations, both in the short- and medium-run.
- Learns the steady state.

Conclusion

- Long-run (**global**) conditions can be replaced with appropriate regularization (**local**) to achieve the optimal solutions.
- The minimum-norm kernels aligns with optimality in economic dynamic models.
- The minimum-norm kernels accurately learn the correct steady state(s).

Conclusion: explicit and implicit Regularization

Machine learning methods:

$$\underbrace{\frac{1}{N}\sum_{i=1}^{N}\mathcal{L}\big(y_i, f_\theta(x_i)\big)}_{\text{loss}} + \underbrace{\lambda\,\Omega(f_\theta)}_{\text{regularization term}}$$

- $f_\theta(\cdot)$: parametric function (kernel methods, neural networks, etc.).
- In this paper, regularization is used to rule out sub-optimal solutions and enforce stability.
- What other roles can regularization play in solving economic models with modern ML methods?

Appendix

Matérn kernel

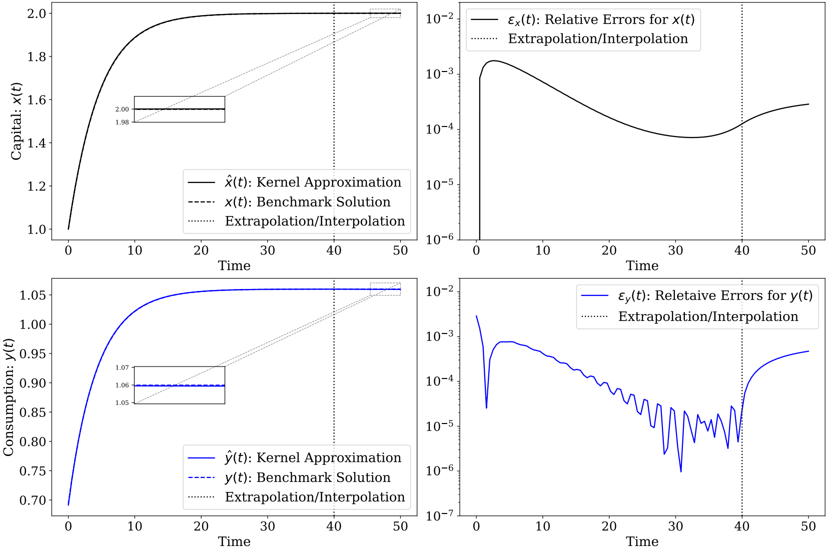
$$K(t, t_j) = C_{\frac{1}{2}}(t, t_j) = \sigma^2 \exp\left(-\frac{|t - t_j|}{\ell}\right),$$

$$K(t, t_j) = C_{\frac{3}{2}}(t, t_j) = \sigma^2 \left(1 + \frac{\sqrt{3}|t - t_j|}{\ell}\right) \exp\left(-\frac{\sqrt{3}|t - t_j|}{\ell}\right),$$

$$K(t, t_j) = C_{\frac{5}{2}}(t, t_j) = \sigma^2 \left(1 + \frac{\sqrt{5}|t - t_j|}{\ell} + \frac{5|t - t_j|^2}{3\ell^2}\right) \exp\left(-\frac{\sqrt{5}|t - t_j|}{\ell}\right).$$

Back

Neoclassical growth: relative errors



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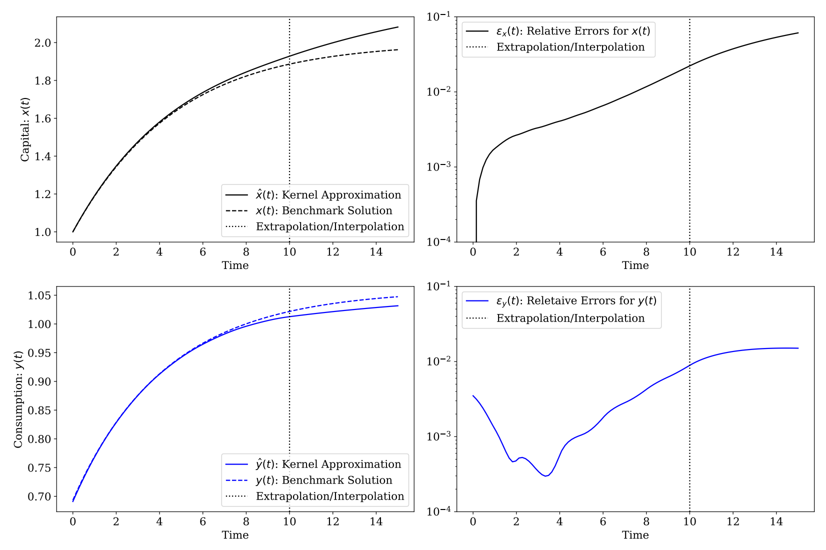
Neoclassical growth: short-run relative errors



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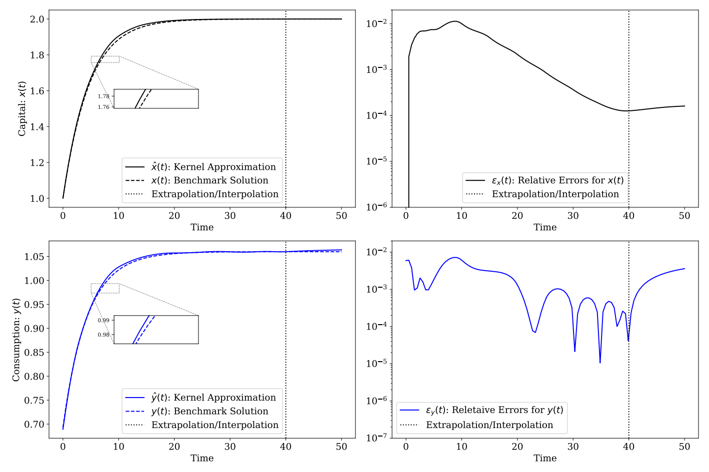
Neoclassical growth: sparse sampling



$$\mathcal{D} = \{0, 1, 3, 5, 10, 15, 20, 25, 30, 35, 38, 40\}$$

Back

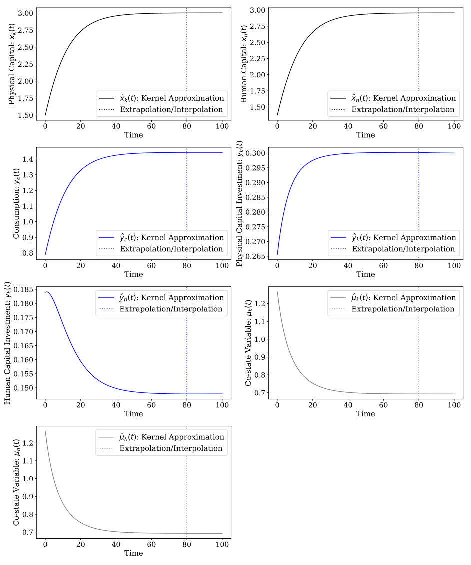
Human capital and growth

$$
\begin{aligned}
\dot{x}_k(t) &= y_k(t) - \delta_k x_k(t), \\
\dot{x}_h(t) &= y_h(t) - \delta_h x_h(t), \\
\dot{\mu}_k(t) &= r\mu_k(t) - \mu_k(t)\left[f_1(x_k(t), x_h(t)) - \delta_k\right], \\
\dot{\mu}_h(t) &= r\mu_h(t) - \mu_h(t)\left[f_2(x_k(t), x_h(t)) - \delta_h\right], \\
0 &= \mu_k(t) y_c(t) - 1, \\
0 &= \mu_k(t) - \mu_h(t), \\
0 &= f(x_k(t), x_h(t)) - y_c(t) - y_k(t) - y_h(t), \\
0 &= \lim_{t\to\infty} e^{-rt} x_k(t)\mu_k(t), \\
0 &= \lim_{t\to\infty} e^{-rt} x_h(t)\mu_h(t),
\end{aligned}
$$

for given initial conditions $x_k(0) = x_{k_0}$, $x_h(0) = x_{h_0}$.

- Human capital is $x_h(t)$, physical capital $x_k(t)$, consumption $y_c(t)$, investment in human capital $y_h(t)$, and investment in physical capital $y_k(t)$, $\mu_k(t)$ and $\mu_h(t)$ are the co-state variables.

Results



Physical Capital: $x_k(t)$
$\hat{x}_k(t)$: Kernel Approximation
Extrapolation/Interpolation
Human Capital: $x_h(t)$
$\hat{x}_h(t)$: Kernel Approximation
Extrapolation/Interpolation
Consumption: $y_c(t)$
$\hat{y}_c(t)$: Kernel Approximation
Extrapolation/Interpolation
Physical Capital Investment: $y_k(t)$
$\hat{y}_k(t)$: Kernel Approximation
Extrapolation/Interpolation
Human Capital Investment: $y_h(t)$
$\hat{y}_h(t)$: Kernel Approximation
Extrapolation/Interpolation
Co-state Variable: $\mu_k(t)$
$\hat{\mu}_k(t)$: Kernel Approximation
Extrapolation/Interpolation
Co-state Variable: $\mu_h(t)$
$\hat{\mu}_h(t)$: Kernel Approximation
Extrapolation/Interpolation
Time