Exploiting Symmetry in High-Dimensional Dynamic Programming

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Motivation

- Most dynamic models in macro (and other fields) deal with either:
  - Representative agent or few agents.
  - A continuum of agents.
- However, many models of interest in macro (IO and trade) deal with **finite** (but large) number of agents
  - Industry dynamics with many firms, agents and industries, even models with networks.
- These models are becoming increasingly popular, **but**:
  - They pose computational challenges as we add more agents.
  - **No (non-heuristic)** algorithm exists providing **global** solutions in the presence of aggregate uncertainty.

Challenges: the curse of dimensionality in equilibrium models

Three components to the curse of dimensionality with many agents (Bellman, 1958, p. IX)

1. The cardinality of the state space is enormous.
    - With 266 state variables, with 2 values per state (zero and one), we have more arrangements (2^{266}) than the estimated number of protons in the universe.
2. With idiosyncratic and aggregate shocks we need to calculate high-dimensional conditional expectations.
3. Finding equilibrium paths to the steady-state (ergodic distributions) are extremely hard in high-dimensions.

Contribution I

Inspired by economic theory, providing novel method for **globally** solving high-dimensional heterogeneous agent models with **aggregate shocks** which relies on:

- A **symmetry** present in many heterogeneous agent models, i.e., **exchangeability** of agents.
  - Example: In general equilibrium models the **Walrasian** auctioneer removes indices.
  - The solution must be faithful to this symmetry caused by this exchangeability of agents.

Contribution II

- **Concentration of measures**, something that resembles the law of large numbers to deal with conditional expectations.
  - More agents makes it easier to forecast the evolution of distributions.
  - Conditional on the aggregate shock, we can use something similar law of large numbers to calculate expectations.
- We show how to implement the symmetry when using **deep neural networks**.

With these we **globally** solve a model with **10,000** agents which was **not possible** before.

Literature Review

- Deep learning as a functional approximation: Maliar et al. (2019), Fernández-Villaverde et al. (2022), Duarte (2018), Azinovic et al. (2022), Han et al. (2021) (a mean-field approach).
- Symmetry in statistics and machine learning: Bloem-Reddy and Teh (2020), Zaheer et al. (2017), and Yarotsky (2018).
- Symmetry in computer science (MDP/RL): Ravindran and Barto (2001) and Narayanamurthy and Ravindran (2008), van der Pol et al. (2020).
- Symmetry in micro and games: Jovanovic and Rosenthal (1988), Hartford et al. (2016)

Background: Deep learning for functional equations

Equilibrium conditions as functional equations

Most theoretical models in economics with equilibrium conditions can be written as functional equations:

- Take some function(s) $\psi \in \Psi$ where $\psi : \mathcal{X} \to \mathcal{Y}$ (e.g. asset price, investment choice, best-response).
- Domain $\mathcal{X}$ could be state (e.g. dividends, capital, opponents state) or time if sequential.
- The "model" is $\ell : \Psi \times \mathcal{X} \to \mathcal{R}$ (e.g., Euler and Bellman residuals, equilibrium FOCs).
- The solution is the root of the model (residuals operator), i.e., $\mathbf{0} \in \mathcal{R}$, at each $x \in \mathcal{X}$.

Then a **solution** is a $\psi^* \in \Psi$ where $\ell(\psi^*, x) = 0$ for all $x \in \mathcal{X}$. How do we find an approximate solution?

Classical solution method for functional equations

Quick **review** of collocation-like methods:

1. **Pick** finite set of D points $\hat{\mathcal{X}} \subset \mathcal{X}$ (e.g., a grid).
2. **Choose** approximation $\hat{\psi}(\cdot;\theta) \in \mathcal{H}(\Theta)$ with coefficients $\Theta \subseteq \mathbb{R}^M$ (e.g., Chebyshev polynomials).
3. **Fit** with nonlinear least-squares
$$\min_{\theta \in \Theta} \sum_{x \in \hat{\mathcal{X}}} \ell(\hat{\psi}(\cdot;\theta), x)^2$$
If $\theta \in \Theta$ is such that $\ell(\hat{\psi}(\cdot;\theta), x) = 0$ for all $x \in \hat{\mathcal{X}}$ we say $\hat{\psi}(\cdot;\theta)$ **interpolates** $\hat{\mathcal{X}}$.
4. The goal is to have good **generalization**:
   - The approximate function is close to the solution outside of $\hat{\mathcal{X}}$.

Deep Neural Networks

Deep learning is **highly-overparameterized** $\mathcal{H}(\Theta)$ ($M \gg D$) designed for good generalization.

- Example: one layer neural network, $\hat{\psi} : \mathbb{R}^Q \to \mathbb{R}$:

$$\hat{\psi}(x;\theta) = W_2 \cdot \sigma\left(W_1 \cdot x + b_1\right) + b_2$$

- $W_1 \in \mathbb{R}^{P\times Q}$, $b_1 \in \mathbb{R}^{P\times 1}$, $W_2 \in \mathbb{R}^{1\times P}$, and $b_2 \in \mathbb{R}$.
- $\sigma(\cdot)$ is a nonlinear function applied element-wise (e.g., $\max\{\cdot, 0\}$).
- $\Theta \equiv \{b_1, W_1, b_2, W_2\}$ are the coefficients, in this example $M = PQ + P + P + 1$.
- Making it "deeper" by adding another "layer": $\hat{\psi}(x;\theta) \equiv W_3 \cdot \sigma(W_2 \cdot \sigma(W_1 \cdot x + b_1) + b_2) + b_3$.
- Very flexible to design $\mathcal{H}(\Theta)$ using economic insights (e.g., encode symmetry).
- Composing $\mathcal{H}(\Theta)$ from multiple function (e.g., deeper) tend to **generalize better** better in practice.

Over-parameterization and convergence

If the number of coefficients is much larger than the number of grid points $M \gg D$, there are many different sets of coefficients that achieve interpolation.

- What is going on?
  - Deep neural networks and their optimizers have an inherent **implicit bias** toward a **unique** class of interpolating solutions.
  - Figuring out this property is a very active field in computer science and optimization theory.
  - Converges to **"simple"** (flat) interpolating functions.
    - They have a built-in **Occam's razor**.
- Modern ML, uses high-dimensional **non-convex** optimizations. Does the initialization of the coefficients matter?

Application

How do we pick our application to show how all this works?

- In terms of application, there are two routes:
  1. Introducing a sophisticated application where the method "shines".
  2. Or, applying it to a well-known example.
- If I tell you about a sophisticated application, how do we know our "solution" method works?
- So we study a well-known example (with a twist).

Our application

A variation of the Lucas and Prescott (1971) model of investment under uncertainty with N firms.

Why?

1. Ljungqvist and Sargent (2018), pp. 226-228, use it to introduce recursive competitive equilibria.
2. Simple model that fits in one slide.
3. Under one parameterization, the model has a known Linear-Quadratic (LQ) solution, which gives us an exact benchmark.
4. By changing one parameter, the model is nonlinear, with **no known** solution. Our method handles the nonlinear case as easily as the LQ case with high accuracy.

Investment under uncertainty

- Industry consisting of $N > 1$ firms, each producing the same good.
- Firm of interest produces output x (x units of capital).
- Thus, the vector $X \equiv [X_1, \dots X_N]^\top$ is the production (or capital) of the whole industry.
- The inverse demand function for the industry is, for some $\nu \geq 1$ (this is our twist):

$$p(X) = 1 - \frac{1}{N} \sum_{i=1}^{N} X_i^\nu$$

- The firm does not consider the impact of its individual decisions on $p(X)$.
- Due to adjustment frictions, investing u has a cost $\frac{\gamma}{2} u^2$.
- Law of motion for capital $x' = (1 - \delta)x + u + \sigma w + \eta\omega$ where $w \sim \mathcal{N}(0, 1)$ an i.i.d. idiosyncratic shock, and $\omega \sim \mathcal{N}(0, 1)$ an i.i.d. aggregate shock, common to all firms.
- The firm chooses u to maximize $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(p(X)x - \frac{\gamma}{2} u^2\right)\right]$.

Recursive problem

The recursive problem of the firm taking the exogenous policy $\hat{u}(\cdot, X)$ for all other firms as given is:

$$v(x, X) = \max_u \left\{ p(X)x - \frac{\gamma}{2}u^2 + \beta \mathbb{E}\left[v(x', X')\right] \right\}$$
$$\text{s.t. } x' = (1-\delta)x + u + \sigma w + \eta\omega$$
$$X_i' = (1-\delta)X_i + \hat{u}(X_i, X) + \sigma W_i + \eta\omega, \quad \text{for } i \in \{1, ..., N\}$$

Take FOCs and equation using standard steps to write equilibrium as the LOM and Euler equation

$$\gamma u(X) = \beta \mathbb{E}\left[p(X') + \gamma(1-\delta)u(X')\right]$$

Curse of dimensionality: a closer look

$$\gamma u(X) = \beta \mathbb{E}\left[p(X') + \gamma(1-\delta)u(X')\right]$$

Let's say there are 1000 **agents**:

1. The domain of the functions of interest $u(\cdot)$ is 1000-**dimensional**.
2. The expectation operator $\mathbb{E}[\cdot]$ is a 1001-**dimensional** integral.
   - 1000 idiosyncratic shocks + 1 aggregate shock.
3. What about the stationarity of the solution and the **transversality condition**?
   - I will come back to this.

Symmetry and Permutation Invariant

General class of problems: A "big X, little x" dynamic programming

$$
\begin{aligned}
v(x, X) &= \max_{u} \left\{ r(x, u, X) + \beta \mathbb{E}\left[v(x', X') \right] \right\} \\
\text{s.t. } x' &= g(x, u) + \sigma w + \eta \omega \\
X' &= G(X) + \Omega W + \eta \omega \mathbf{1}_N
\end{aligned}
$$

1. x is the individual state of the agent.
2. X is a vector stacking the individual states of all of the N agents in the economy.
3. u is the control variable.
4. w is random innovation to the individual state, stacked in $W \sim \mathcal{N}(\mathbf{0}_N, \mathbf{I}_N)$ and where, w.l.o.g., $w = W_1$.
5. $\omega \sim \mathcal{N}(0, 1)$ is a random aggregate innovation to all the individual states.

Permutation Groups

- A permutation matrix is a square matrix with a single 1 in each row and column and zeros everywhere else.
- Let $\mathcal{S}_N$ be the set of all $n!$ permutation matrices of size $N \times N$. For example:

$$S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

- Multiplying vector $v \in \mathbb{R}^N$ by $\pi \in S_N$ reorders elements of v

Permutation-invariant dynamic programming

Definition

A 'big X, little x' dynamic programming problem is a **permutation-invariant dynamic programming problem** if, for all $(x, X) \in \mathbb{R}^{N+1}$ and all permutations $\pi \in \mathcal{S}_N$

1. The reward function r is **permutation invariant**:

$$r(x, u, \pi X) = r(x, u, X)$$

2. The deterministic component of the law of motion for X is **permutation equivariant**:

$$G(\pi X) = \pi G(X)$$

3. The covariance matrix of the idiosyncratic shocks satisfies

$$\pi \Omega = \Omega \pi$$

Permutation invariance of the optimal solution

Proposition

The optimal solution of a permutation-invariant dynamic programming problem is permutation invariant. That is, for all $\pi \in \mathcal{S}_N$*:*

$$u(x, \pi X) = u(x, X)$$

and:

$$v(x, \pi X) = v(x, X)$$

Can $u(x, X)$ permutation invariance guide $\mathcal{H}(\Theta)$ choice?

Representation of permutation-invariant functions

Proposition

(based on Wagstaff et al., 2019) Let $u : \mathbb{R}^{N+1} \to \mathbb{R}$ *be a continuous permutation-invariant function under* $\mathcal{S}_N$*, i.e., for all* $(x, X) \in \mathbb{R}^{N+1}$ *and all* $\pi \in \mathcal{S}_N$*:*

$$u(x, \pi X) = u(x, X)$$

Then, there exist a latent dimension $L \leq N$ *and continuous functions* $\rho : \mathbb{R}^{L+1} \to \mathbb{R}$ *and* $\phi : \mathbb{R} \to \mathbb{R}^L$ *such that:*

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}\phi(X_i)\right)$$

Representation of permutation-invariant functions: Discussion and intuition

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}\phi_1(X_i), \cdots, \frac{1}{N}\sum_{i=1}^{N}\phi_L(X_i)\right)$$

- This proposition should remind you of Krusell-Smith (1998), $L = 1$, $\phi(X_i) = X_i$.
- Key benefit for approximation is the **representation** (ρ, ϕ).
- Fitting a ρ and ϕ rather than f directly leads to **far better generalization** on $\mathcal{X}$. Why?:
  - Imposing structure on $\mathcal{H}(\Theta)$, functions that know a lot about the economic problem.
- In practice: $L \ll N$ generalizes very well. Regression Example

High-dimensional expectation: concentration of measures

High-dimensional expectation

Euler's equation:

$$\beta \mathbb{E}\left[p(X') + \gamma(1-\delta)u(X')\right]$$

- Linear example: $p(X') = 1 - \frac{1}{N}\sum_{i=1}^{N} X_i'$.
- Conditioned on the aggregate shock ω, **law of large numbers**:

$$\mathbb{E}\left[p(X') \mid \omega\right] \approx p(X')|\omega, \quad \text{for large } N.$$

- In general, can we say the same thing about $u(X')$?

$$\mathbb{E}\left[u(X') \mid \omega\right] \approx u(X')|\omega, \quad \text{for large } N?$$

- **Yes**, but $u(\cdot)$ has to satisfy some properties.

Expected gradient bounded in N

Definition (Expected gradient bounded in N)

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a bounded function in N and $z \sim \mathcal{N}(\mathbf{0}_N, \mathbf{I}_N)$ be a normalized Gaussian random vector. The function f has its expected gradient bounded in N if there exists a C such that:

$$\mathbb{E}\left[\|\nabla f(z)\|^2\right] \leq \frac{C}{N},$$

where C does not depend on N.

$$\mathbb{E}_W\left[\|\nabla u(X')\|^2\right] \leq \frac{C}{N}$$

- The policy to be well-behaved (non-explosive gradients).
- Other agent's influence vanishes.

Main result II: Concentration of measure

Proposition

Suppose $z \sim \mathcal{N}(\mathbf{0}_N, \Sigma)$, where the spectral radius of Σ, denoted by $\rho(\Sigma)$, is independent of N, z^1 a draw from z, and $f : \mathbb{R}^N \to \mathbb{R}$ is a function with expected gradient bounded in N. Then:

$$\mathbb{P}\left(\left|f(z^1) - \mathbb{E}\left[f(z)\right]\right| \geq \epsilon\right) \leq \frac{\rho(\Sigma)C}{\epsilon^2}\frac{1}{N}$$

- As Ledoux (2001) puts it: "A random variable that depends in a Lipschitz way on many independent variables (but not too much on any of them) is essentially constant."
- With concentration of measure, dimensionality is not a curse; it is a blessing.

Implication: We can calculate $\mathbb{E}_W[u(X')|\omega]$ with a *single draw* of idiosyncratic shocks W:

- $\mathbb{E}_W[u(X')|\omega] \approx u(X')|\omega$.
- Reducing an $N+1$-dimensional conditional expectation to a 1-D one (with good approximation).

Summarizing results

- The structure symmetry imposes on the functions leads to better **generalization**
  - Functions extrapolate better outside of the grid points $\hat{\mathcal{X}}$.
- Concentration of measures provides a fast method for calculating the conditional expectations.
  - Calculate with **one draw** of the idiosyncratic shocks (conditional on the aggregate shock).

Solving the Model

Design of $\mathcal{H}(\Theta)$: Deep learning architectures

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}\phi(X_i)\right)$$

Three cases for ϕ:

- Identity function: One moment $\rightarrow \phi(\text{Identity})$.

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}X_i\right)$$

- Up to degree four polynomials: 4 moments $\rightarrow \phi(\text{Moments})$.

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}X_i, \cdots, \frac{1}{N}\sum_{i=1}^{N}X_i^4\right)$$

Design of $\mathcal{H}(\Theta)$: Deep learning architectures

- A deep neural network for ϕ, with $L = 4 \rightarrow \phi(\text{ReLU})$.

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}\phi_1(X_i), \cdots, \frac{1}{N}\sum_{i=1}^{N}\phi_4(X_i)\right)$$

If polynomials for ϕ: A finite set of moments à la Krusell-Smith.

- In all cases, ρ is a highly over-parameterized neural network with four layers.
- The baseline $\phi(\text{Identity})$, $\phi(\text{Moments})$, and $\phi(\text{ReLU})$ have $49.4K$, $49.8K$, and $66.8K$ coefficients.

Solution method follows "interpolation" methods

1. **Pick**: $\hat{\mathcal{X}}$ as simulated trajectories from X_0:
   - Only need 100 to 1000 points regardless of dimensionality of the state space N.
   - Because we use economic insight, i.e., symmetry which gives us good generalization.
2. **Choose**: Design the $\mathcal{H}(\Theta)$ with ρ and ϕ as discussed:
   - ϕ(Identity), ϕ(Moments), and ϕ(ReLU).

Using concentration of measures:

- **One draw** $\hat{W} = \{\hat{W}_1, \ldots, \hat{W}_N\}$ of the idiosyncratic shocks. For a given $u(\cdot;\theta)$, and aggregate shock ω calculate:

$$X_i' = (1-\delta)X_i + u(X) + \sigma\hat{W}_i + \eta\omega, \quad \text{for } i \in \{1, ..., N\}.$$

Solution method follows "interpolation" methods

- Approximate the Euler residuals

$$\varepsilon\left(X; u(\cdot;\theta)\right) \equiv \gamma u(X;\theta) - \beta \mathbb{E}\left[P(X') + \gamma(1-\delta)u(X';\theta)\right]$$

and a quadrature method for aggregate shocks. ⏩ error analysis in N

3. **Fit**: The residuals $\varepsilon\left(X; u(\cdot;\theta)\right)$, that is the "model" i.e., ℓ.

$$\min_{\theta\in\Theta} \sum_{X\in\hat{\mathcal{X}}} \varepsilon\left(X; \hat{u}(\cdot;\theta)\right)^2$$

4. **How to Verify/Test**: Given the approximate solution simulate new paths from X_0 and check the Euler residuals (ε).

Study two cases: linear ($\nu = 1$) and nonlinear ($\nu > 1$) demand functions

Case 1: Linear to verify algorithms and methods

- With $\nu = 1$, we have a linear demand function: $p(X) = 1 - \frac{1}{N}\sum_{i=1}^{N} X_i$.
- It generates a Linear-Quadratic (LQ) dynamic programming problem (only the mean of X_i matters).
- We can find the exact $u(x, X)$, LQ has algebraic solutions.
- The LQ solution gives us a benchmark against which we can compare our deep learning solution.
- The neural network figures out very quickly that the solution is $u(x, X) = H_0 + \frac{1}{N} H_1 \sum_{i=1}^{N} X_i$ and finds a high-dimensional approximation which matches that for the training grid.

Equilibrium Paths: Linear case

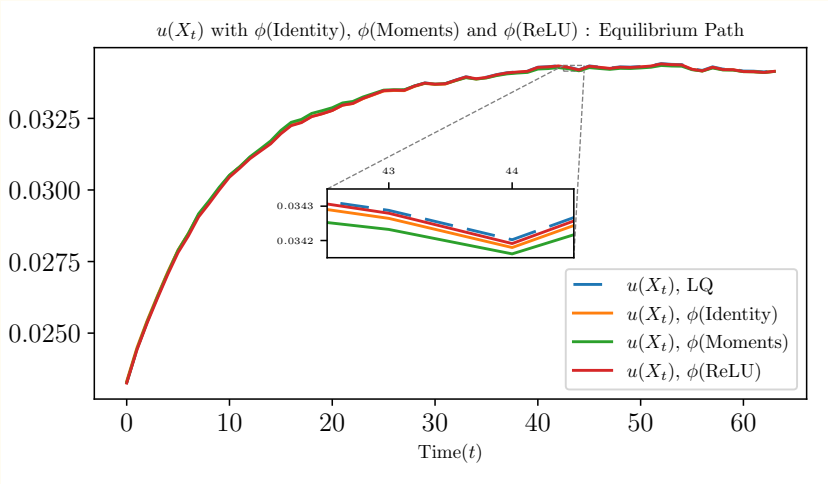


Figure 1: Comparison between baseline approximate solutions and the LQ solution for the case with $\nu = 1$ and $N = 128$.

Accuracy of Solutions: Linear case

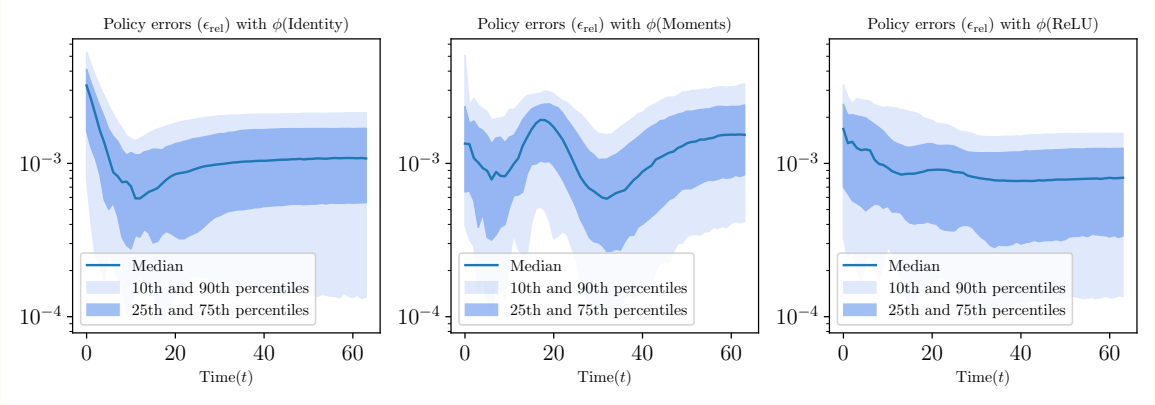


Figure 2: The relative errors for $\nu = 1$ and $N = 128$ for ϕ(Identity), ϕ(Moments), and ϕ(ReLU). The dark blue curve shows the average residuals along equilibrium paths for 256 different trajectories.

Computation time: Linear case

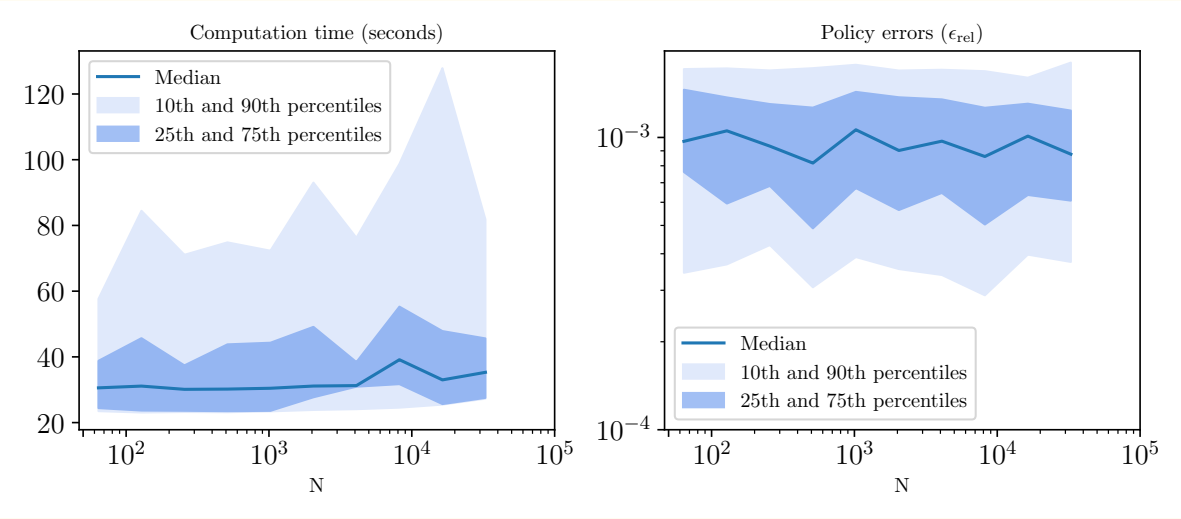


Figure 3: Performance of the ϕ(ReLU) for different N (median value of 21 trials).

Case 2: Nonlinear case with no "closed-form" solution

- With $\nu > 1$, we have a nonlinear demand function: $p(X) = 1 - \frac{1}{N}\sum_{i=1}^{N} X_i^{\nu}$.
- Now, the whole distribution of X_i matters.
- But we can still find the solution to this nonlinear case using exactly the same functional approximation and algorithm as before.
- We do not need change anything in the code except the value of ν.
- We do not have an exact solution to use as a benchmark, but can check residuals.
- Same model and method. Computation time by N nearly the same to linear case
  - Link to the code

Euler residuals: Nonlinear case

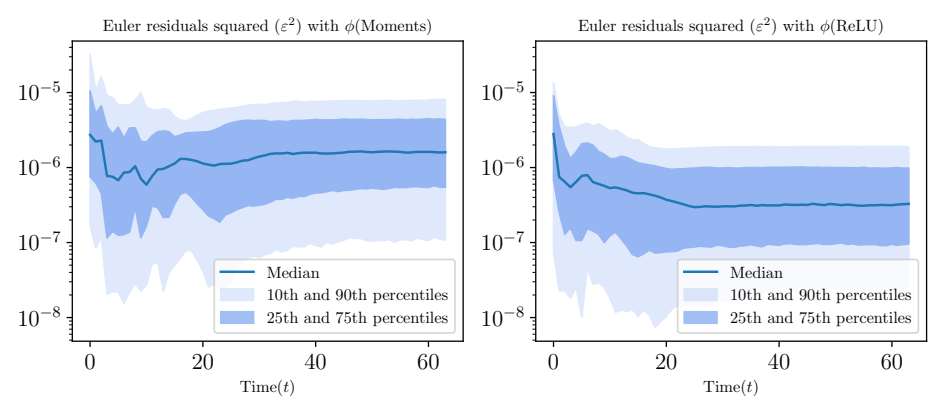


Figure 4: The Euler residuals for $\nu = 1.5$ and $N = 128$ for ϕ(Moments) and ϕ(ReLU). The dark blue curve shows the average residuals along equilibrium paths for 256 different trajectories. The shaded areas depict the 2.5th and 97.5th percentiles.

Equilibrium paths: Nonlinear case

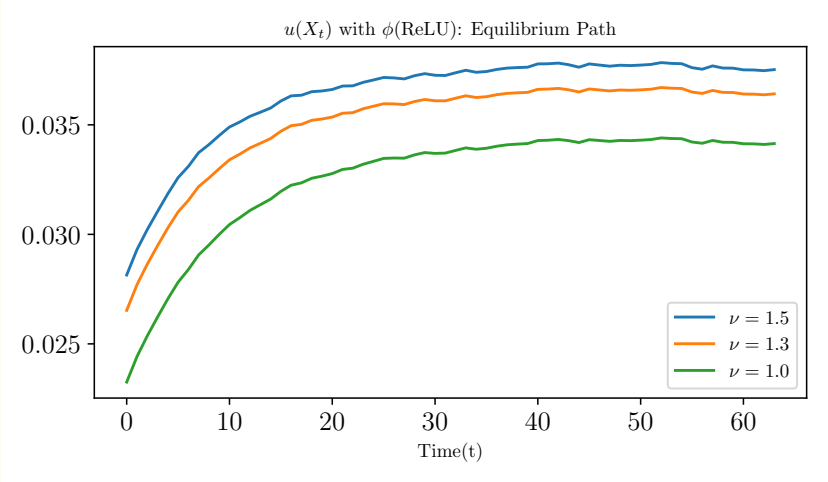


Figure 5: The optimal policy u along the equilibrium paths for $\nu = [1.0, 1.05, 1.1, 1.5]$ and $N = 128$. Each path shows the optimal policy for a single trajectory.

Some challenging question: Generalization puzzle

Question I: Generalization and overfitting

- From statistical learning and numerical analysis we know:
  - More coefficients in the family of parametric functions $\mathcal{H}(\Theta)$ leads to over-fitting and poor generalization (bias-variance trade-off).
  - We have $70K$ parameters, and $< 1K$ grid points.
  - The results indicate the opposite: More coefficients $\rightarrow$ better generalization.

How come we achieve great generalization?

Some challenging questions: Multiplicity and transversality puzzle

Question II: Multiplicity and transversality

$$\gamma u(X) = \beta \mathbb{E}\left[p(X') + \gamma(1-\delta)u(X')\right]$$
$$X_i' = (1-\delta)X_i + u(X) + \sigma W_i + \eta\omega, \quad \text{for } i \in \{1, ..., N\}$$

with linear prices. Guess and verify with $u(X) \equiv H_0 + \frac{1}{N} H_1 \sum_{i=1}^{N} X_i$

- The Euler equation is quadratic $\rightarrow$ **two** solutions: $(H_0^-, H_1^-), (H_0^+, H_1^+)$:
  - $H_1^- < 0 \rightarrow$ **stationary** solution, $H_1^+ > 0 \rightarrow$ **non-stationary** solution.
  - We have **no explicit** device in our algorithm to weed out the second solution.

How come we never observe the non-stationary solution in the results?

Understanding the **implicit bias** of deep neural networks answers both questions. Ebrahimi Kahou et al. 2024, I&II address these two challenging questions.

Implicit bias, Generalization, and Stationarity.

Representation with linear prices

Recall the representation,

$$u(x, X) = \rho\left(x, \frac{1}{N}\sum_{i=1}^{N}\phi(X_i)\right).$$

For $\nu = 1$ we can show that the following **exact solution** holds with our representation

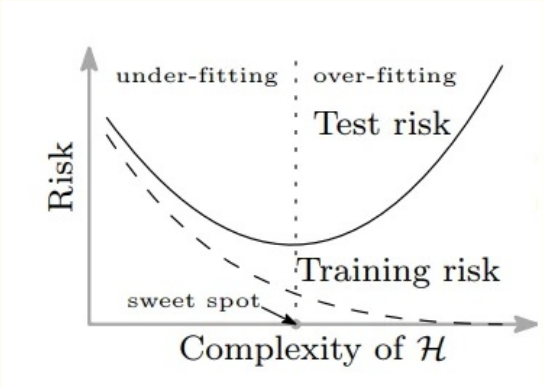
- $\phi(X_n) = X_n$ identity, $L = 1$.
- $\rho(x, y) = \theta_1 + \theta_2 y$.
- Doesn't matter how to generate X since only need 2 points.
- Let's do it with 3 points.

Extreme example of generalizability of neural networks

- Forget we know any closed form, and see if over-fitting hurts us.
- Fit **three** grid points in $\mathbb{R}^{512}$ (an economy with $N = 512$ agents).
- Flexible functional form with **17.7 K** coefficients.
- Now, evaluate for a whole bunch of reasonable trajectories from the initial condition and check the policy error:
  - 5×10^{-5} MSE of Euler, approximately **0.06%** relative error of $u(X)$.

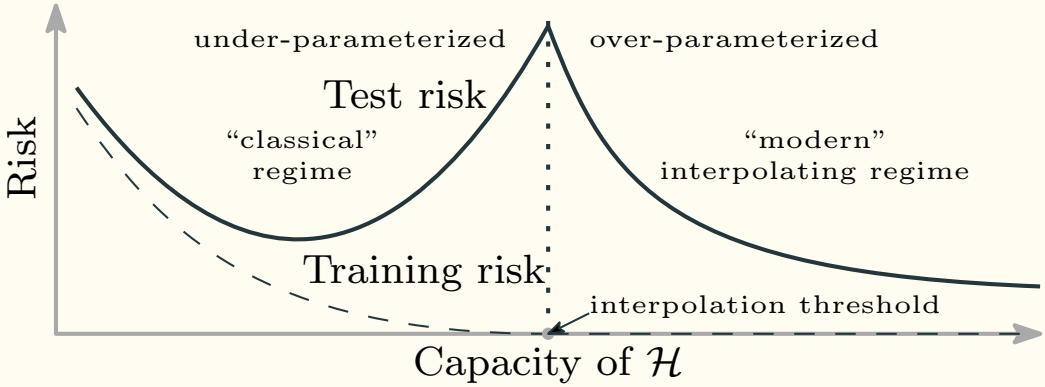
This is related to a literature called **double descent**.

Classical bias-variance trade-off



- Complexity of $\mathcal{H}$: think of
  - Degree of Chebyshev polynomials, polynomials, ...
  - Coefficients of the neural networks (weights and biases)

The cure to over-fitting is to add more parameters



Belkin et al., 2019: Traditional statistics/bias-variance trade-off stop around the interpolation threshold.

Deep learning optimizes in a space of functions

Remember

$$\min_{\theta \in \Theta} \sum_{x \in \hat{\mathcal{X}}} \ell(\hat{\psi}(\cdot; \theta), x)^2$$

- Deep learning: number of coefficients is much larger than the number of grid points.
- Since $M \gg D$, it is possible for $\hat{\psi}$ to interpolate and the objective value will be ≈ 0.
- Since $M \gg D$ there are many solutions (e.g., θ_1 and θ_2),
  - Agree on the grid points: $\hat{\psi}(x; \theta_1) \approx \hat{\psi}(x; \theta_2)$ for $x \in \hat{\mathcal{X}}$.
- Since individual θ are irrelevant it is helpful to think of optimization directly within $\mathcal{H}$

$$\min_{\hat{\psi} \in \mathcal{H}} \sum_{x \in \hat{\mathcal{X}}} \ell(\hat{\psi}, x)^2$$

But which $\hat{\psi}$?

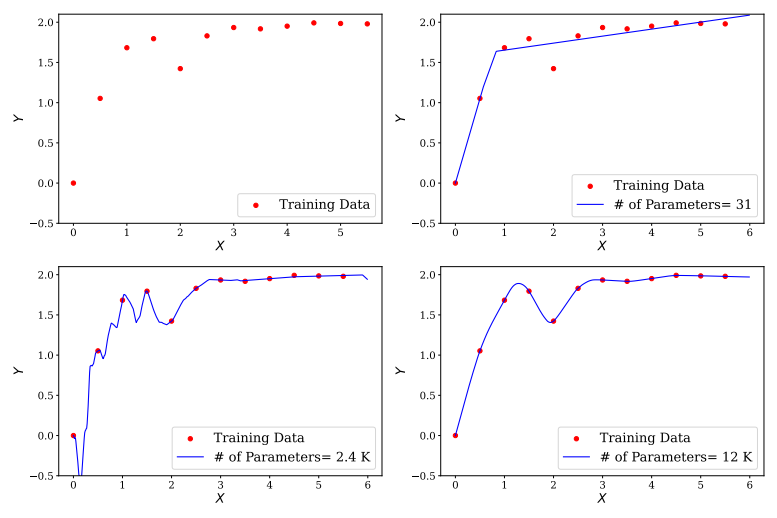
Deep learning and interpolation

- For M large enough, optimizers **tend to** converge to **unique "simple"** $\hat{\psi}$ (w.r.t to some norm $\|\cdot\|_S$). Unique both in $\hat{\mathcal{X}}$ and $\mathcal{X}$. There is a **bias** toward a specific class of solutions.
- **How to interpret:** interpolating solutions for some functional norm $\|\cdot\|_S$

$$\min_{\hat{\psi}\in\mathcal{H}} ||\hat{\psi}||_S$$
$$\text{s.t. } \ell(\hat{\psi}, x) = 0, \quad \text{for } x \in \hat{\mathcal{X}}$$

  - Comp Sci literature refers to this as the **inductive bias** or **implicit bias**: optimization process is biased toward particular $\hat{\psi}$.
  - Small values of $\|\cdot\|_S$ corresponds to **flat** solutions with **small gradients** (w.r.t. input).

Flat and smooth interpolation: illustration



Answering the challenging questions

- Answering **generalization puzzle**: Flat interpolation leads to good generalization:
  - If the true underlying functions is flat between (and outside) the points.
  - The cure to over-fitting is to add more parameters.
- Answering **multiplicity puzzle**: In the linear set-up, the explosive solution has larger derivatives (less flat) than the non-explosive one i.e, $|H_1^+| > |H_1^-|$:
  - The deep-learning based solution **automatically** satisfies stationarity.
- Ebrahimi Kahou et al. 2024, I& II explore this for many more dynamic models in macroeconomics (e.g., neoclassical growth and asset pricing) we show:
  - We can have short- and medium-run accurate solutions (even in non-stationary cases) .
  - We dont need to calculate the steady-state (ergodic distribution).

Conclusions

Extensions

1. Decreasing returns to scale: the policy becomes a function of x.
2. Multiple productivity types (e.g., two different groups).
3. Complex idiosyncratic states (e.g., a agent is described with more than one variable).

Summarizing our contribution

- **Method** for solving **high-dimensional** dynamic programming problems and competitive equilibria with idiosyncratic and aggregate shocks relying
  - Symmetry.
  - Concentration of measures: Dimensionality is a **blessing** not a curse.
- Using **economic theory** (i.e., exchangeability) and **deep learning** for function approximation with a huge # of parameters ($\gg$ grid points)
  - Achieve great generalization: key to alleviate the curse of dimensionality.
- Implementation
  - Can deal with 10000+ agents.
  - Can deal with 10000+ dimensional expectations with one Monte-carlo draw.

Future challenges ahead

- The role of sampling in high-dimensional spaces.
- Equilibrium selection, in case of multiplicity.
- Guaranteed stable solutions.

Appendix

Example: Symmetry in a regression problem

Data generating process:

$$y = f(X) = \left(\frac{1}{N} \sum_{i=1}^{N} x_i^{\zeta} \right)^{\frac{1}{\zeta}}$$

- Pick $a \sim \mathcal{U}(a_{\min}, a_{\max})$
- Draw $X = (x_1, \cdots, x_N) \sim \mathcal{N}(a, \sigma^2)$: M data points
- $\zeta = 1.5$, $a_{\min} = 1$, $a_{\max} = 2$, $\sigma = 0.2$, $M = 10$
- Generate a set of new data and look at the mean of the relative errors

$$\left| \frac{\hat{f}(X) - f(X)}{f(X)} \right|$$

Results: Symmetry in a regression problem

Table 1: Results for the regression problem

$\hat{L}$ / N	1	2	3	4	5	20	NONE
2	1.85%	1.19%	1.91%	1.27%	1.7%	1.75%	1.36%
4	1.04%	1.7%	1.76%	0.76%	0.97%	0.83%	1.65%
8	0.57%	0.57%	0.71%	0.64%	0.6%	0.56%	3.68%
16	0.5%	0.49%	0.51%	0.45%	0.42%	0.48%	4.91%
32	0.32%	0.35%	0.32%	0.28%	0.37%	0.51%	5.69%
64	0.32%	0.27%	0.27%	0.33%	0.28%	0.29%	5.77%
128	0.32%	0.22%	0.24%	0.28%	0.23%	0.22%	5.77%

back

Definition (Bounded functions in N)

Let:

$$\mathcal{L}(M) \equiv \{y \in \mathbb{R}^N : |y_i| \leq M \ \forall i = 1, \ldots, N\}$$

be an N-dimensional hypercube in $\mathbb{R}^N$. A function $f : \mathbb{R}^N \to \mathbb{R}$ is bounded in N if for every M there exists K_M such that

$$\sup_{y \in \mathcal{L}(M)} |f(y)| < K_M,$$

where K_M is a constant that does not depend on N, but may depend on M.

- Example $f(y) = \frac{1}{N} \sum_{i=1}^{N} y_i \to \sup_{y \in \mathcal{L}(M)} |f(y)| < M$.
- To avoid $f(y) = \sum_{i=1}^{N} y_i \to \sup_{y \in \mathcal{L}(M)} |f(y)| < NM$.

back

Concentration of measure is the bless of dimensionality

In the linear case we know the closed form solution for u

$$\hat{\varepsilon}(X; u) - 0 \sim \mathcal{N}\left(0, \frac{\sigma_\varepsilon^2}{N}\right)$$
$$u(\hat{X}') - \mathbb{E}\left[u(X') \mid \omega\right] \sim \mathcal{N}\left(0, \frac{\sigma_u^2}{N}\right)$$

- Conditional expectation becomes constant as N gets large.
  - One **single Monte-carlo draw** of the idiosyncratic shocks is enough.

back

Analytic euler error due to the concentration of measure

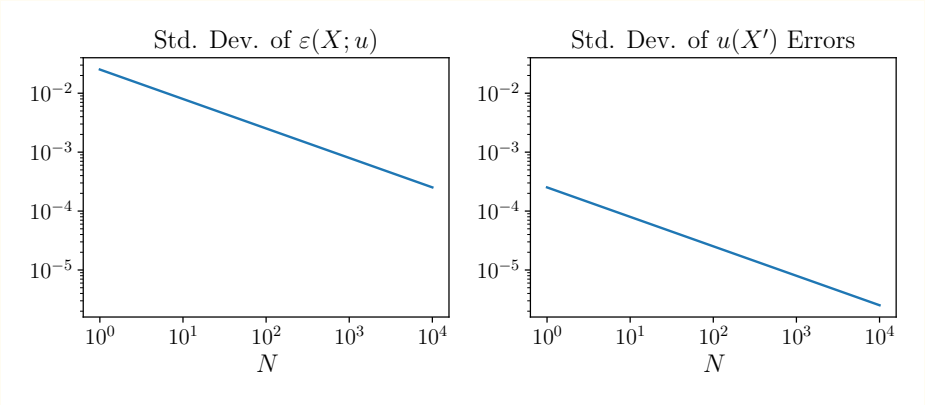


Table 2: Performance of Different Networks in Solving the Linear Model

Group	Description	Success (%)	Parameters (Thousands, K)	Time (s)	Train MSE (ε)	Val MSE (ε)	Test MSE (ε)	Policy Error (ϵ_{rel})
Identity	Baseline	48%	49.4	28	6.7e-07	4.9e-07	5.0e-07	0.10%
	Baseline: Moments (1,2,3,4)	59%	50.3	36	9.0e-07	8.7e-07	1.2e-06	0.13%
Moments	Moments (1,2)	54%	50.0	33	1.0e-06	8.7e-07	1.0e-06	0.12%
	Very Shallow (1 layer)	0%	0.8	-	-	-	-	-
	Baseline: L= 4	97%	199.6	17	1.3e-06	3.6e-07	3.6e-07	0.09%
	L = 2	93%	201.1	17	1.3e-06	4.0e-07	4.3e-07	0.09%
Deep Sets	L = 16	93%	204.2	14	1.5e-06	3.5e-07	3.5e-07	0.10%
	Deep (ϕ : 2 layers, ρ : 4 layers)	100%	215.9	25	2.0e-06	3.8e-07	3.7e-07	0.10%
	Shallow (ϕ : 1 layer, ρ : 2 layers)	1%	68.0	16	1.6e-07	3.3e-07	3.5e-07	0.10%

Table 3: Nonlinear Model Performance

group	description	Time (s)	Params (K)	Train MSE (ε)	Test MSE (ε)	Val MSE (ε)
ϕ(Moments)	Baseline	26	49.8	6.0e-06	5.0e-06	3.8e-06
	Moments (1,2)	27	49.5	8.0e-06	5.1e-06	3.6e-06
	Very Shallow (1 layer)	252	**0.6**	8.3e-06	**1.4e+00**	5.0e-06
	Thin (32 nodes)	66	3.2	1.1e-05	9.7e-06	4.4e-06
ϕ(ReLU)	Baseline	60	67.1	1.4e-05	4.7e-06	3.3e-06
	L = 8	73	68.1	1.1e-05	4.9e-06	2.0e-06
	L = 16	72	70.2	1.5e-05	5.4e-06	1.7e-06
	Very Shallow (ϕ, ρ : 1 layer)	136	**1.4**	8.9e-06	**4.8e+06**	4.9e-06
	Shallow (ϕ, ρ: 2 layers)	47	34.3	1.0e-05	9.2e-06	2.8e-06
	Thin (ϕ, ρ :32 nodes)	52	4.5	1.3e-05	6.0e-06	2.7e-06

Parameters

- $\gamma = 90$, $\beta = 0.95$, $\sigma = 0.005$, $\eta = 0.001$.