The Blessings of Overparameterization: Applications in Solving Economic Models

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What economists mean by “solving a model”

An economic model is a system of functional equations

- Economists build models by specifying how agents behave: what they want (preferences), what they can do (technology), and what limits them (constraints).

- Optimizing behavior implies **functional equations whose unknowns are functions**.

- **Example:** a social planner chooses a consumption rate c_t given capital k_t and stochastic productivity z_t. Capital evolves as:

$$\dot{k}_t = e^{z_t} k_t^{\alpha} - \delta k_t - c_t$$

  Output $e^{z_t} k_t^{\alpha}$ is produced, capital depreciates at rate δ, and whatever is not consumed is invested.

- Productivity follows an Ornstein–Uhlenbeck process:

$$dz_t = -\mu z_t \, dt + \sigma \, dW_t$$

- The social planner maximizes expected discounted lifetime utility:

$$\max_{\{c_t\}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) \, dt$$

What does "solving" this model mean?

- The problem has a recursive representation in the form of a PDE, the **Hamilton–Jacobi–Bellman equation**:

$$\boxed{\rho V = \max_c \left\{ u(c) + V_k[e^z k^\alpha - \delta k - c] - \mu z\, V_z + \tfrac{1}{2}\sigma^2 V_{zz} \right\}}$$

- The unknown is the **value function** $V(k, z)$:
  1. The first-order condition $u'(c) = V_k(k, z)$ gives the optimal policy $c = (u')^{-1}(V_k)$.
  2. From the capital equation, the drift is $\dot{k} = e^z k^\alpha - \delta k - (u')^{-1}(V_k)$.
- "Solving the model" means finding $V(\cdot)$ that satisfies the HJB equation over the state space.
- In general, no closed-form expression exists.
- An alternative is to solve the Euler equation:

$$\frac{d\,[u'(c_t)]}{u'(c_t)} = \left[\rho - \left(\alpha e^{z_t} k_t^{\alpha-1} - \delta\right)\right] dt + \sigma\, \frac{u''(c_t)}{u'(c_t)}\, d_z(k_t, z_t)\, dW_t$$

The mathematical structure you already know

- Rewrite the HJB equation as:

$$\mathcal{T}(V) \equiv \rho V - \max_c \left\{u(c) + V_k[e^z k^\alpha - \delta k] - \mu z\, V_z + \tfrac{1}{2}\sigma^2 V_{zz}\right\} = 0$$

- Strip away the economics, and you are looking for a function V such that:

$$\boxed{\mathcal{T}(V) = \mathbf{0}}$$

  where $\mathcal{T}$ is a nonlinear operator between function spaces. Nothing exotic here.
- The HJB equation is a second-order nonlinear PDE. You have seen these in optimal control and differential games.
- The classical numerical approach is exactly what you would do:
  1. Propose a parameterized approximation $\widehat{V}_\theta$.
  2. Evaluate the residual $\mathcal{R}[\widehat{V}_\theta](X)$ at a collection of points.
  3. Find θ that drives the residual to zero.

So what makes the economics problem different?

- When you solve a typical PDE, you know the equation before you write a single line of code. The coefficients, the forcing term, the boundary conditions: all given.
- You can pick your grid, choose your discretization, and refine your mesh. The equation never changes on you. Better numerics give a better approximation to a **fixed target**.
- In economics, this is not the case. The law of motion for the state variables is the policy function we are trying to find. Agents observe the current state, make a choice, and that choice determines where the system goes next.
- So improving your approximation changes the dynamics, which changes which parts of the state space the system visits, which changes where your approximation needs to be accurate.

The difference in one sentence

In typical natural science problems, the dynamics are an input. In economics, the dynamics are the output.

The equilibrium loop

- Return to the HJB equation:

$$\rho V = u\big((u')^{-1}(V_k)\big) + V_k\big[e^z k^\alpha - \delta k - (u')^{-1}(V_k)\big] - \mu z\, V_z + \tfrac{1}{2}\sigma^2 V_{zz}$$

- We already mentioned this is a second-order nonlinear PDE.
- But look at the drift term for k: $e^z k^\alpha - \delta k - (u')^{-1}(V_k)$.
- The law of motion for the state depends on V, the very function we are solving for!
- **The equilibrium loop:** the "coefficients" of the PDE are endogenous to the optimal choices of the agents.

Why it matters

- Since the drift depends on V, we need to choose where to evaluate the residual.
- A natural choice: weighting the residual by the **ergodic distribution** ξ^*:

$$\theta^* = \arg\min_{\theta} \int \mathcal{R}[\widehat{V}_\theta](X)^2 \, d\xi^*(X)$$

- But ξ^* is generated by the law of motion, which depends on V, the function we are solving for:

$$V \longrightarrow \text{drift of } k \longrightarrow \xi^* \longrightarrow \text{evaluation points} \longrightarrow \widehat{V}_\theta \longrightarrow V$$

- Improving $\widehat{V}_\theta$ changes the drift, changes ξ^*, and changes where the nodes should have been. Moving the net moves the target.

Practical fix

Iterate between sampling and optimization. Simulate from the current $\widehat{V}_\theta$, recompute evaluation points, re-optimize. Convergence is not guaranteed but works well in practice.

Why traditional methods hit a wall

- When the state space is low-dimensional ($N \leq 4$), everything we just described works fine: lay down a grid or use Chebyshev collocation and solve.
- But many of the questions economists want to answer require large N:
  - Heterogeneous households differing in wealth, income, age, and portfolio composition.
  - Firms interacting strategically in a dynamic oligopoly.
  - Spatial models with many regions linked by trade and migration.
- Grid-based methods need m^N points. Chebyshev collocation with tensor products needs D^N terms. Both become infeasible beyond $N = 5$ or so.
- Perturbation (Taylor expansion around steady state) sidesteps the grid but is only locally accurate. It fails with kinks, occasionally binding constraints, or large shocks.

Enter neural networks (the part you know)

- The idea is simple: replace the fixed polynomial basis with a neural network:

$$\boxed{\widehat{d}_\theta(X) = f_\theta(X), \qquad \theta^* = \arg\min_\theta \frac{1}{L}\sum_{l=1}^{L} \mathcal{R}[f_\theta](X_l)^2}$$

- You already know why this helps: universal approximation, dimension-free rates for the Barron class (Barron, 1993), learned representations. I will take such background as given.
- What is specific to our setting:
  1. The residual $\mathcal{R}$ is not a supervised-learning loss. It comes from the model's equilibrium conditions (Bellman, Euler, market clearing).
  2. Evaluation points $\{X_l\}$ are regenerated each epoch from the current approximation (the equilibrium loop we just discussed).
  3. There may be additional constraints: transversality conditions, non-negativity of consumption, budget balance.

The blessings of overparameterization

The question this paper asks

- Now that you know the setup (finding functions that satisfy operator equations, with the complication that the sampling measure is endogenous), a practical question arises:
- **How large should the network be?**
- Standard statistical wisdom says: more parameters than evaluation points $\Rightarrow$ overfitting $\Rightarrow$ poor out-of-grid performance.
- If true here, the practitioner must carefully tune network width, adding a fragile hyperparameter choice to an already demanding computational problem.

Thesis

This paper argues the opposite. In the context of solving these operator equations, overparameterization is a **blessing**. It improves both accuracy and, more importantly, algorithmic stability.

Two findings

- **Finding 1, no overfitting:** as the number of network parameters grows, out-of-grid residuals (HJB and Euler) decrease. The approximate value and policy functions converge toward their benchmark counterparts.
- **Finding 2, algorithmic stability:** with small networks, solutions obtained from different random initializations of θ disagree substantially.
  - The researcher has **no way of knowing** which initialization produced the better solution. The "answer" depends on the random seed.
  - As network width increases, this disagreement collapses: 50 independent random seeds converge to the same function.

Practical implication

When in doubt, use a wider network. Overparameterization is not a source of risk to be managed but a feature to be exploited.

Connection to the machine learning literature

- Our findings connect to a broader phenomenon in modern machine learning.

Double descent

Belkin et al. (2019): the classical bias–variance tradeoff breaks down for overparameterized models. Test error can decrease again after an initial overfitting peak.

Benign overfitting

Bartlett et al. (2020): minimum-norm interpolants can generalize well even when parameters vastly exceed observations.

- **Our contribution:** this benign behavior extends to function approximation problems in computational economics, where the loss is defined by **equilibrium conditions** (HJB, Bellman, or Euler equations) rather than by labeled data.

Contrast with perturbation methods

- A central challenge in solving nonlinear models by perturbation: higher-order Taylor expansions generate explosive sample paths.
- Standard fix: **pruning** (Kim et al., 2008; Andreasen et al., 2018). Cross-products of higher-order terms are deliberately dropped to restore stability.
- Pruning is an exercise in deliberate under-parameterization: parameters are removed to make the approximation well-behaved.
- Our paper delivers the **opposite message**: in the neural network approach, increasing network width is what delivers accuracy and stability.

Slogan

The perturbation literature prunes to survive; neural network solvers grow to thrive.

Experimental design

- We demonstrate these properties across three canonical models with known benchmarks:

1. Linear-Quadratic
Closed-form solution. Sharpest test of approximation error.

2. McCall Job Search
1D state, kink at reservation wage. Non-smooth approximation.

3. Real Business Cycle
2D stochastic state. No closed form. Benchmarked vs. VFI.

- In all three: accuracy and stability improve with width. No sign of overfitting.
- Theorem? So far, just a conjecture.

Network architecture and training protocol

- **Hidden width** H varied systematically.
- One hidden layer: $f_\theta(x) = W^{(2)}\sigma\big(W^{(1)}x + b^{(1)}\big) + b^{(2)}$, with $H(n+2)+1$ parameters.
- Two hidden layers: $f_\theta(x) = W^{(3)}\sigma\big(W^{(2)}\sigma\big(W^{(1)}x + b^{(1)}\big) + b^{(2)}\big) + b^{(3)}$, with $H(n+H+3)+1$ parameters.
- **Activation functions:** ReLU $\sigma(z) = \max\{0, z\}$ (baseline), Leaky ReLU $\sigma(z) = \max\{\alpha z, z\}$ with $\alpha = 0.01$, and Sigmoid $\sigma(z) = 1/(1+e^{-z})$.
- **Training:** minimize squared residual loss (Euler or Bellman) using Adam.
- **Key design choice:** 50 independent random seeds per specification. This lets us measure both accuracy (median across seeds) and stability (width of the 10th–90th percentile band).

Application 1: Linear-Quadratic regulator

The LQ problem

- The general deterministic LQ problem:

$$v(x) = \max_u \left\{ -x^\top R\, x - u^\top Q\, u + \beta\, v(x') \right\}$$
$$\text{s.t.} \quad x' = Ax + Bu, \quad x_0 \text{ given}$$

- Closed-form solution:

$$\boxed{v(x) = -x^\top P x, \qquad u(x) = -Fx}$$

  where P and F are obtained from the discrete-time algebraic Riccati equation.
- Sharpest test: approximation error can be computed exactly at every point.

A competitive firm with adjustment costs

- A price-taking firm in a competitive industry.
- Own output y, aggregate output Y (exogenous). Inverse demand: $\alpha_0 - \alpha_1 Y$.
- The firm chooses investment u subject to quadratic adjustment cost $\frac{\gamma}{2}u^2$:

$$v(y, Y) = \max_u \left\{ (\alpha_0 - \alpha_1 Y)y - \tfrac{\gamma}{2}u^2 + \beta\, v(y', Y') \right\}$$

$$\text{s.t.} \quad Y' = h_0 + h_1 Y, \quad y' = y + u$$

- State is two-dimensional: (y, Y). With augmented state $x = (1, y, Y)^\top$, this maps into the general LQ framework.
- Euler equation:

$$\gamma\, u(y, Y) = \beta \left[\gamma\, u(y', Y') + (\alpha_0 - \alpha_1 Y') \right]$$

Parameterization

Description	Symbol	Value	
Discount factor	β	0.9	
Demand intercept	α_0	1.0	inverse demand level
Demand slope	α_1	1.3	price sensitivity to aggregate
Adjustment cost	γ	100	convex investment cost
Aggregate drift	h_0	0.05	constant in $Y' = h_0 + h_1 Y$
Aggregate persistence	h_1	0.9	AR coefficient for Y

LQ: deep learning implementation

- **Architecture:** one-hidden-layer ReLU networks for value and policy functions. Hidden units $H = 1$ to $H = 7$ (4 to 22 parameters). 50 seeds per configuration.
- **Training data:** the optimal u depends only on Y (y enters linearly and does not affect the marginal return to investment), so we minimize the mean squared Euler residual over $n_Y = 6$ grid points:

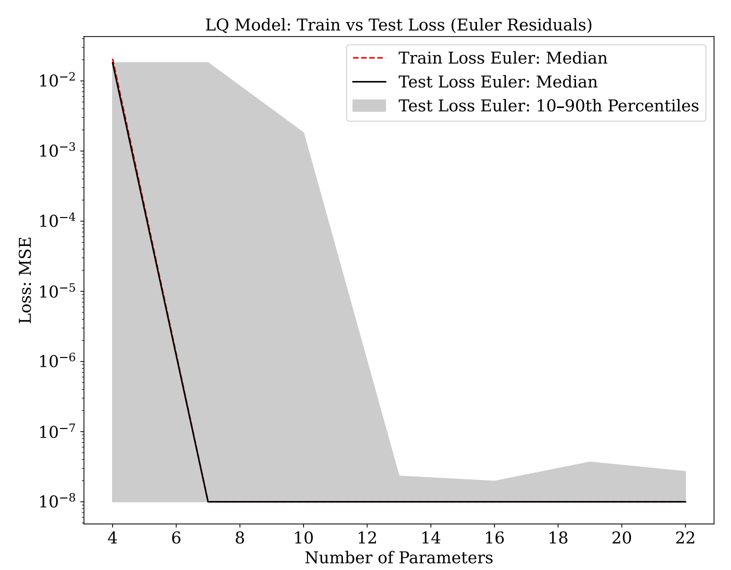
$$\boxed{\min_{\theta_u} \frac{1}{n_Y} \sum_{i=1}^{n_Y} \left[\gamma\, u(Y_i; \theta_u) - \beta\big(\gamma\, u(Y_i'; \theta_u) + \alpha_0 - \alpha_1 Y_i'\big) \right]^2}$$

  The value function is then trained on a 12×12 Cartesian grid of (y, Y) pairs. Adam optimizer with early stopping at loss $< 10^{-8}$.
- **Test data:** $T = 29$ period path simulated from fixed initial conditions. Test points lie in the interior of the state space and never on a training node.

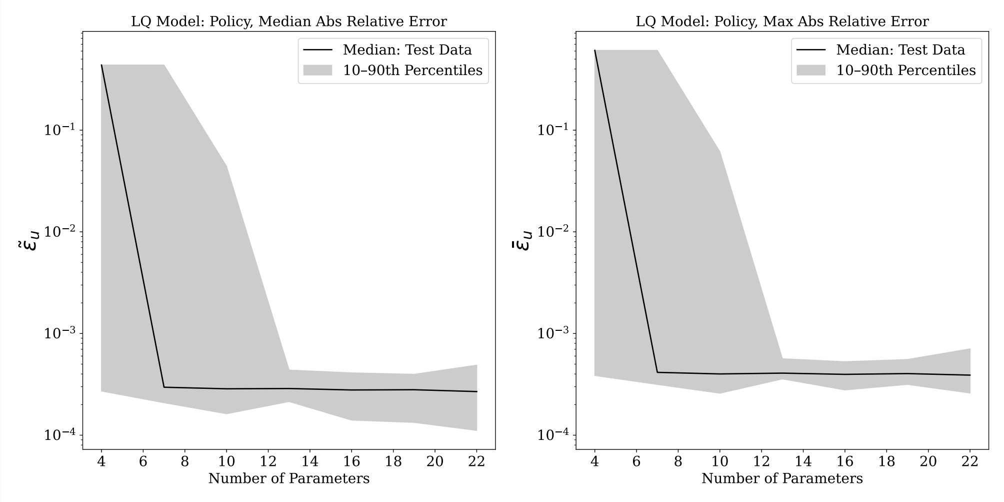
LQ: training and test data

- Red dots: 12×12 training grid. Blue crosses: $T = 29$ test path simulated from fixed initial conditions.
- Test points lie strictly in the interior of the state space and never coincide with a training node.

LQ: train and test Euler-residual loss

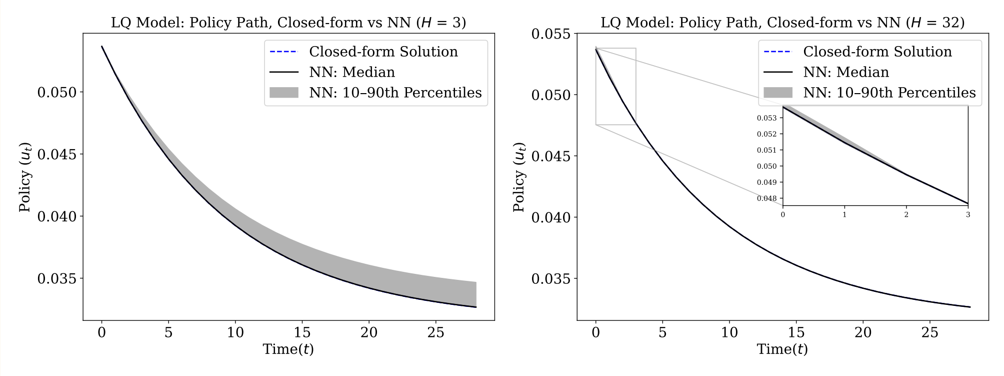


LQ: absolute relative error vs. closed-form solution



- Median (left) and maximum (right) absolute relative error of the NN policy. Both statistics and their

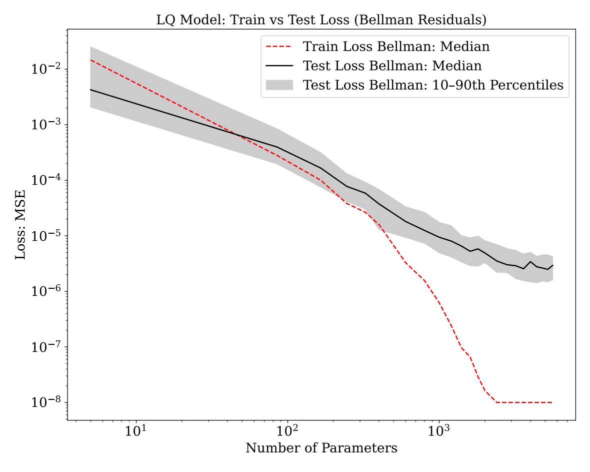
LQ: policy paths, underparameterized vs. overparameterized



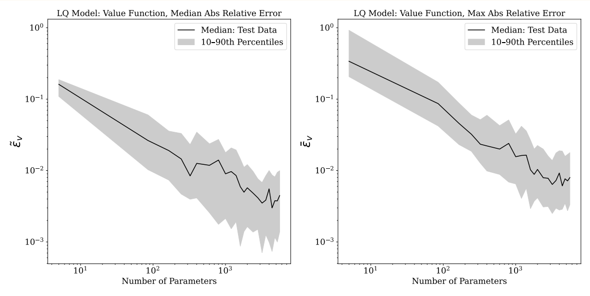
- Left: $H = 3$ (underparameterized). Median is roughly accurate, but the across-seed band is wide.
- Right: $H = 32$ (overparameterized). Median tracks the closed-form solution closely and the band collapses.

LQ: value function results

(A) Train and test Bellman MSE



(B) Median and max relative error



- Value function network (trained conditional on estimated policy) displays the same pattern: accuracy

Application 2: McCall job-search model

From a linear to a non-smooth operator equation

- The LQ model is linear, which makes it the cleanest test but also the easiest.
- Does the blessing survive when the operator $\mathcal{T}$ is nonlinear and the solution is non-smooth?
- The McCall model (McCall, 1970) provides exactly this test: the value function has a **kink** at the reservation wage, so the network must approximate a non-differentiable function.
- An unemployed worker receives i.i.d. wage offers $w \sim f$ on $[0, B]$ each period.
- Accept: permanent income stream $w/(1-\beta)$. Reject: unemployment compensation c plus continuation value.
- Bellman equation (a nonlinear integral equation with a max operator):

$$\boxed{v(w) = \max\left\{\frac{w}{1-\beta},\ c + \beta \int_0^B v(w')f(w')\,dw'\right\}}$$

McCall: the kink and why it matters

- Optimal policy: threshold form. There exists a reservation wage $\bar{w}$ such that accept iff $w > \bar{w}$.
- Closed-form value function:

$$v(w) = \begin{cases} \bar{w}/(1-\beta) & \text{if } w \leq \bar{w}, \\ w/(1-\beta) & \text{if } w > \bar{w}. \end{cases}$$

- The reservation wage $\bar{w}$ solves:

$$\bar{w} - c = \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w}) f(w')\, dw'$$

  LHS: net gain from rejecting the marginal offer. RHS: expected gain from continued search.

- The kink at $\bar{w}$ is a demanding test: the network must learn a non-smooth function from only $N = 12$ training points.

Parameterization

Description	Symbol	Value	
Discount factor	β	0.9	
Wage support	B	1.0	upper bound of $\mathrm{U}[0, B]$
Unemployment benefit	c	0.1	flow payoff while searching

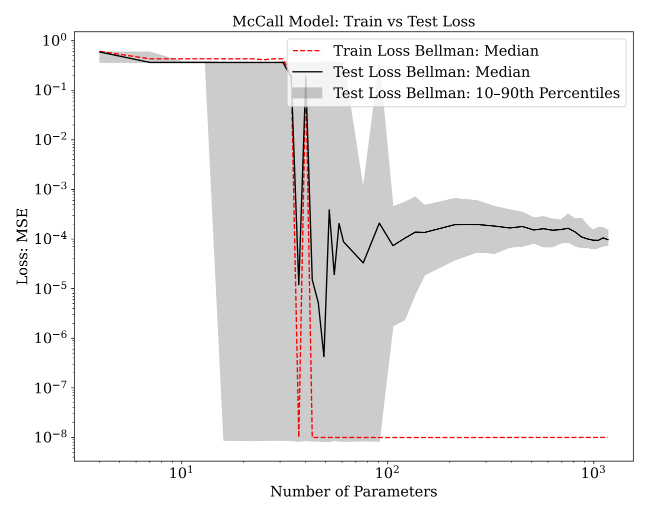
McCall: deep learning implementation

- **Architecture:** one-hidden-layer ReLU network $v(w;\theta_v)$, linear output. 50 random seeds per width.
- **Training:** $N = 12$ uniform grid on $[0,1]$. Minimize mean squared Bellman residual:

$$\boxed{\min_{\theta_v} \frac{1}{N} \sum_{w_i \in \mathcal{D}} \left[v(w_i;\theta_v) - \max\left\{ \frac{w_i}{1-\beta},\ c + \beta \int_0^B v(w';\theta_v) f(w')\, dw' \right\} \right]^2}$$

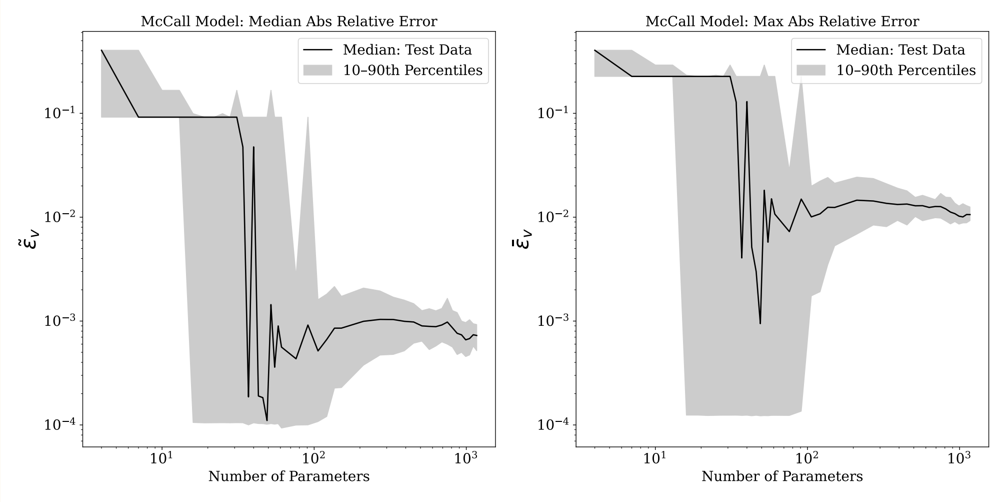
  Integral by quadrature. Adam with early stopping at 10^{-8}.
- **Test:** finer $N = 100$ uniform grid, strictly outside the training set.
- **Accuracy metric:** absolute relative error $\varepsilon_v(w) = |v(w;\theta_v^*) - v(w)|/|v(w)|$. Summarized by max and median across test grid, then 10th/50th/90th percentiles across 50 seeds.

McCall: train and test Bellman-residual loss



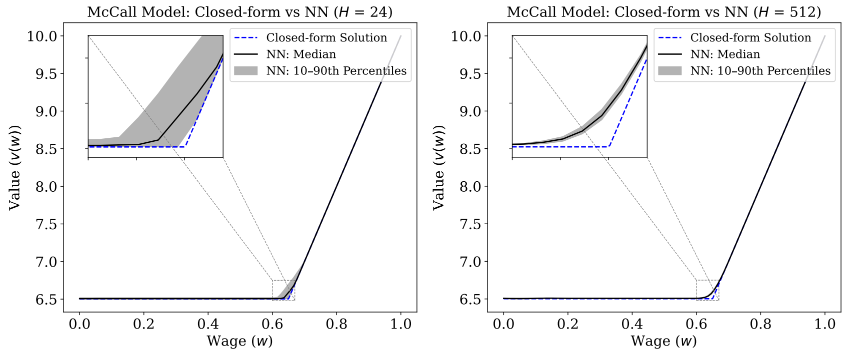
- Train loss hits the early-stopping floor at the interpolation threshold ($\sim$12 parameters, matching $N = 12$ training points).
- Test loss descends again in the overparameterized regime: the **double-descent pattern**.
- The across-seed band collapses.

McCall: absolute relative error



- Median (left) and max (right) relative error fall from $\sim$30% to $\sim$2% as the network enters the

McCall: value function, underparameterized vs. overparameterized



- Left: $H = 24$ (underparameterized). Median tracks the closed-form solution well, but the across-seed band is wide.
- Right: $H = 512$ (overparameterized). Band is nearly invisible: 50 random initializations converge to the same function.

Application 3: Real Business Cycle model

From deterministic to stochastic, from closed form to none

- The LQ model is linear with a closed-form solution. The McCall model is nonlinear with a kink but still one-dimensional and still admits a closed form.
- The Real Business Cycle model (Kydland and Prescott, 1982) is the most demanding test: a two-dimensional stochastic state space (k_t, z_t) and no analytical solution.
- Social planner, Cobb–Douglas production $e^{z_t} k_t^\alpha$, stochastic TFP:

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad c_t + k_{t+1} = e^{z_t} k_t^\alpha + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,1)$$

- Key object: policy function $k'(k, z)$. Benchmark: value function iteration (VFI) on a fine grid.

RBC: the Euler equation

- Euler equation:

$$\boxed{u'(c_t) = \beta\,\mathbb{E}_t\left[u'(c_{t+1})\left(\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} + 1 - \delta\right)\right]}$$

- **LHS**: marginal utility cost of saving one unit today.
- **RHS**: expected discounted marginal utility gain from extra capital tomorrow (marginal product plus undepreciated capital).
- Nonlinearity plus stochastic two-dimensional state means no analytical solution.
- This is the operator equation from slide 1, now in its full stochastic form.

Parameterization

Description	Symbol	Value	
Discount factor	β	0.96	
Capital share	α	1/3	Cobb–Douglas exponent
Depreciation rate	δ	0.1	per-period capital depreciation
TFP persistence	ρ	0.9	AR(1) coefficient for z_t
TFP volatility	σ	0.01	std. dev. of ε_t
Utility function	$u(c)$	$\ln c$	log utility

RBC: deep learning implementation

- **Architecture:** one-hidden-layer ReLU network $k'(k, z; \theta)$. Output layer: Softplus activation $\log(1 + e^x)$ to enforce $k' > 0$. 50 seeds per width.
- **Training:** 20×20 Cartesian grid ($N = 400$). Euler residual:

$$\boxed{\ell_{\text{Euler}}(k, z; \theta) = \frac{1}{c(k, z; \theta)} - \beta \sum_{i=1}^{M} \frac{w_i}{\sqrt{\pi}} \left[\frac{\alpha e^{\rho z + \sqrt{2}\sigma\zeta_i} k'(k, z; \theta)^{\alpha - 1} + 1 - \delta}{c\big(k'(k, z; \theta),\ \rho z + \sqrt{2}\sigma\zeta_i;\ \theta\big)} \right]}$$

  with $M = 15$ Gauss–Hermite nodes. Adam with StepLR scheduler.
- **Penalty term:** $\lambda\big(k'(k^*, 0; \theta) - k^*\big)^2$ with $\lambda = 0.01$, enforcing a necessary implication of the transversality condition at the steady state.

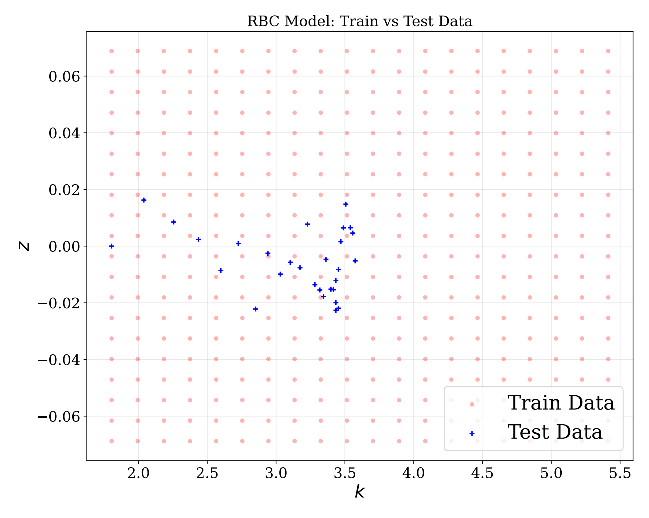
RBC: why the penalty term?

- The Euler equation is a necessary but not sufficient condition for optimality.
- Both the stable and explosive solution manifolds through the steady state satisfy it identically.
- What rules out explosive solutions is the transversality condition (TVC):

$$\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0$$

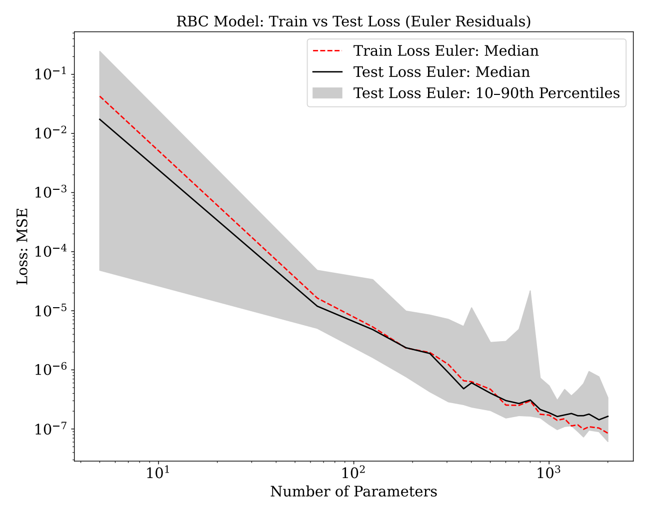
- The Euler residual loss does not enforce the TVC directly.
- The penalty resolves this multiplicity by imposing $k'(k^*, 0) = k^*$: the steady state is a fixed point of the policy function.
- Without this, gradient-based training can converge to the explosive manifold: small Euler residuals but divergent simulated paths.

RBC: training and test data



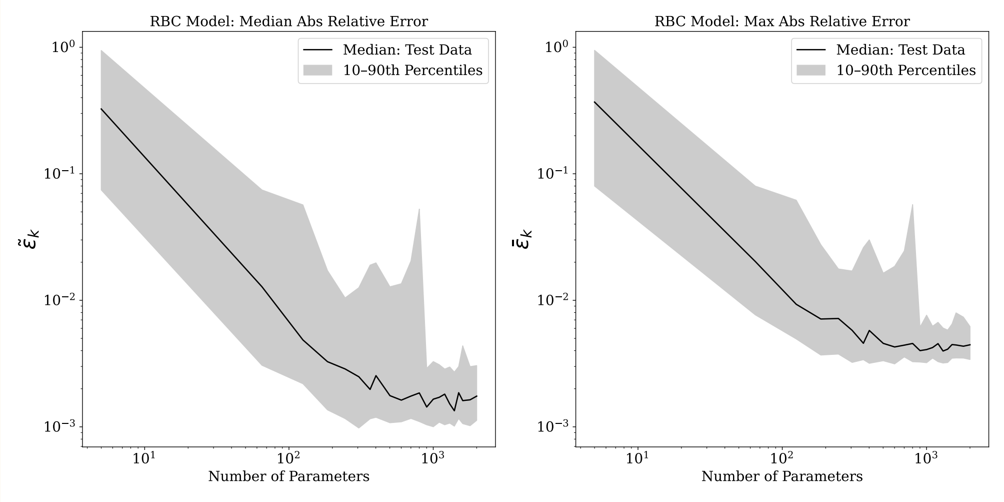
- Red dots: 20×20 training grid.
- Blue crosses: $T = 29$ path simulated under the VFI policy starting at $k_0 = 0.5\,k^*$.
- Test path visits the interior of the state space and is never on a training node.

RBC: train and test Euler-residual loss



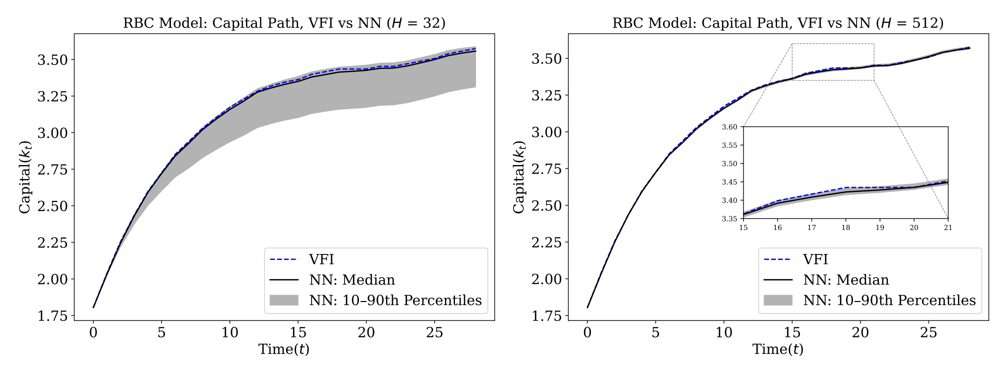
- Both losses decline as the network grows wider, with no **sign of overfitting**.
- The across-seed band narrows sharply: overparameterization simultaneously improves accuracy and reduces initialization sensitivity.

RBC: absolute relative error vs. VFI



- Median (left) and maximum (right) absolute relative error of the NN capital policy versus VFI.

RBC: capital paths, underparameterized vs. overparameterized



- Left: $H = 32$. Median path close to VFI, but across-seed band is wide. Accurate on average but unreliable.
- Right: $H = 512$. Median tracks VFI almost exactly over the full horizon <u>and</u> the band collapses to near zero.

Robustness

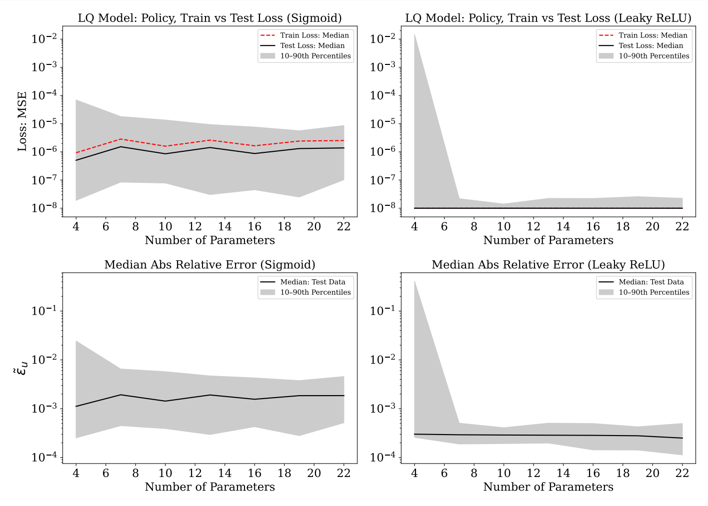
Are these results specific to ReLU and one hidden layer?

- A natural concern: the blessing might depend on the particular activation function or the single-layer architecture.
- We test robustness along two dimensions:
  - **Activation functions:** Sigmoid and Leaky ReLU in addition to the baseline ReLU.
  - **Network depth:** two-hidden-layer architectures in addition to the baseline one-hidden-layer.

Result

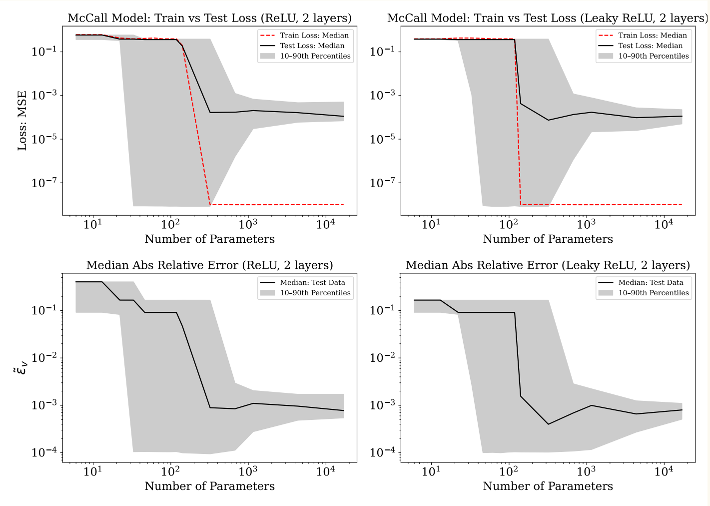
The blessing holds in every configuration we tried. The next slides document this for each model.

Robustness: alternative activations for the LQ model



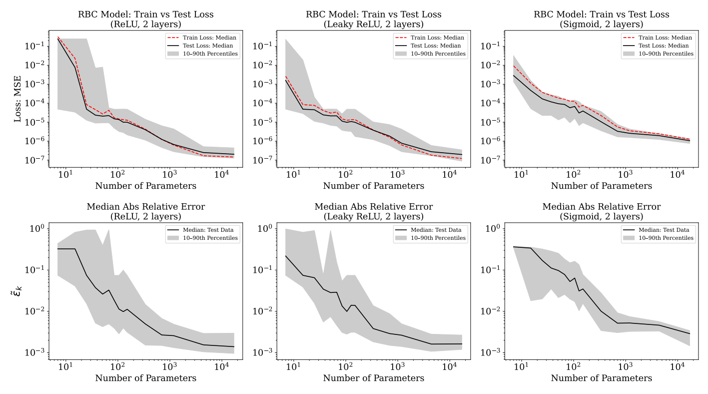
- Sigmoid (left) and Leaky ReLU (right).
- Top row: train/test Euler MSE. Bottom row: median relative error.
- Both activations replicate the main finding: accuracy improves with width, across-seed dispersion collapses.

Robustness: deeper networks for the McCall model



- Two-hidden-layer architecture with ReLU (left) and Leaky ReLU (right).
- Quantitatively similar results: the blessing is not specific to one-hidden-layer networks.

Robustness: deeper networks for the RBC model



- Two-hidden-layer architecture with ReLU (left), Leaky ReLU (center), Sigmoid (right).
- All three activations replicate the main finding for a stochastic model with no closed-form solution.

Robustness: summary

- The blessing of overparameterization is robust across all dimensions most relevant to applied practice:
    - **Activation functions:** ReLU, Leaky ReLU, Sigmoid.
    - **Network depth:** one and two hidden layers.
    - **Operator type:** Euler residual (LQ, RBC), Bellman residual (McCall).
    - **Solution regularity:** smooth (LQ), non-smooth with kink (McCall), stochastic without closed form (RBC).
    - **State dimensionality:** 1D (McCall), 2D (LQ, RBC).

Conclusion

Takeaways

- **Finding 1:** no overfitting. As network width increases, out-of-sample Euler and Bellman residuals decline. Value and policy functions converge toward their benchmarks.
- **Finding 2:** algorithmic stability. Small networks produce solutions that depend on the random seed. Wide networks concentrate across seeds, making the algorithm reliable and initialization-independent.
- Documented consistently across three operator equations: linear with closed form (LQ), nonlinear with kink (McCall), stochastic with no analytical solution (RBC).
- Robust across activations (ReLU, Leaky ReLU, Sigmoid) and depths (one and two hidden layers).

Bottom line

When in doubt, use a wider network. Overparameterization is a blessing, not a curse.

Appendix

Two PDEs, one key difference

- The time-dependent **Schrödinger equation** for a single particle in one dimension:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi$$

Unknown: $\psi(x,t)$. The potential $V(x)$ is given.

- The **Hamilton–Jacobi–Bellman equation** for the stochastic neoclassical growth model:

$$\rho V = \max_c \left\{u(c) + V_k[e^z k^\alpha - \delta k - c] - \mu z\, V_z + \tfrac{1}{2}\sigma^2 V_{zz}\right\}$$

Unknown: $V(k,z)$. The drift of k depends on $(u')^{-1}(V_k)$, a derivative of the unknown.