Optimal Entry Decision with Correlated Variable Cost and Output Price

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Abstract. In models with irrecoverable investment and uncertainty in the output price it is a well-established result that uncertainty increases the output price that a firm starts investment. This paper studies a model of irrecoverable investment (entry) where the variable cost and output price are characterized by two correlated geometric Brownian motions. The numerical results indicate that in the presence of high levels of correlation the impact of uncertainty in output price is ambiguous and depends on the level of variable cost. Specifically, increasing uncertainty in output prices increases the entry output price for low levels of variable cost and the reverse happens for high levels of variable cost. Therefore, in the presence of high levels of correlation the conventional result does not hold anymore. Moreover, this study establishes that increasing the correlation level decreases the entry output price.

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1. Introduction

Understanding firms' investment decision is one of the most crucial and active fields of inquiry in economics. This problem manifests itself in many fields of economics such as economic growth, industrial organization, public, and financial economics. As noted by Bloom (2014) and observed by Alexopoulos and Cohen (2009) uncertainty significantly affects firms' decision to invest, and hire labor. One crucial question for academics and policy-makers is the relation between the uncertainty in market conditions and a firm's investment choice. The uncertainty can stem from different sources such as the uncertainty in the firm's output price, input cost (marginal cost), and competition level.

Many investment problems with uncertainty require an initial irrecoverable expenditures which is sometimes referred to as *sunk cost*. The sunk cost can be thought of as a market or industry specific capital cost. For instance, it could be the lease cost of a firm that requires office space for operating when there is high cost of cancellation. The sunk cost could be in the form of capital depreciation. Consider an artificial intelligence consulting company that requires high performance computing power for its operation. However, due to technological progress these computers substantially lose their market value after couple of years. In many cases a firm has an option to postpone investment[1]. Due to the dynamic and stochastic nature of this problems delaying investment decision affect a firm's profit in two opposite ways. For instance consider a firm that has the option of investing and entering a market with stochastic output price. If the firm delays the investment decision, it loses the revenue of not operating (opportunity cost of not operating). However, delaying provides the firm an opportunity of having new information about the output price and entering the market when the price is high, see Bernanke (1983) for a detailed treatment this trade-off. Pindyck (1990) provides a comprehensive review of the problems with sunk cost and option to delay investment.

One of the most influential models of investment (entry/exit) with irrecoverable cost and option to delay is proposed by Dixit (1989). In this model a firm has the option of entering a market by paying a sunk cost. When the firm is active the output price follows a geometric Brownian motion, and each unit of output costs a deterministic variable cost. This uncertainty in the output price could be thought of stochastic shifts in the demand for the firm's product. The model establishes that higher levels uncertainty in output price increases the value of waiting (postponing). Considering that the price is always positive, and zero is an absorbing barrier of the output price, the firm's revenue is bounded from below. Therefore, it is optimal for the firm to wait longer and enter the market at a higher

[1]However, there are cases that postponing investment is not feasible or optimal. For instance, when a firm has to patent a product, or lunch a new product before its competitors.

levels of output price.[2]

Most of the models of investment with uncertainty only consider one source of uncertainty, for instance uncertainty in output prices or input prices. This assumption significantly reduces the mathematical complexity of these problems. However, it severely restricts the applicability of these models to real world problems of investment. For instance, Marmer and Slade (2018) empirically study the effect of uncertainty on investment decision for the case of U.S copper mines. They make a puzzling observation, that the firms' (mines') scrap value upon exit is almost the same as the sunk cost , i.e. the firm's gets most of the initial sunk cost recovered by liquefying the mine. This observation is a manifestation of simplistic assumptions in investment models under uncertainty and their limited applicability to real world investment problem.

This paper introduces a model of investment with more than one source of uncertainty. Specifically, this research studies the entry decision of a firm with a sunk cost and the option to delay where the variable cost and output prices are characterized by two correlated geometric Brownian motions. For the sake of simplicity, it is assumed that when investment (entry) decision is made then exit is not an option.[3] One can think of several reasons to explain the correlation between the variable cost and output price. One important example of correlated input and output price can be found in the literature of foreign direct investment, if a product is produced in one country and sold in another the correlation between the input and output is lower than the situation if they were both produced in the same country. See McGrattan (2012) for an introduction to the theory and some evidence on foreign direct investment.

From an economic perspective there is a rich literature that studies the effect of uncertainty in market condition on entry and investment decision, both empirically and theoretically. To name a few, Caballero (1991) studies the effect of adjustment costs and competition level on entry decision. Bar-Ilan and Strange (1996) studies how time-to-build can hasten investment under uncertainty. Grenadier and Wang (2007) studies the effect of time-inconsistent preferences on entry decision. Paddock, Siegel, and Smith (1988) show that uncertainty increases the value of offshore oil leases which consequently hastens investment decision. However, to the best of the author's knowledge the only manuscript that studies the correlation between variable cost and output prices as a generalization of

[2]Mathematically speaking, the value function of waiting is a convex function of the output price, therefore by Jensen's inequality uncertainty in output price increases the value of waiting. The same effect happens in McCall job search models, that reservation wage that an unemployed worker above which accepts the job offer increases with the uncertainty of wage distribution, see McCall (1970) and Ljungqvist and Sargent (2018).

[3]The numerical solution used in this study has the flexibility to allow exit by paying some exit cost f.

Dixit (1989) is provided by Bar-Ilan and Borodko (2019).

The key difference between this research and Bar-Ilan and Borodko (2019) lies in the structure of sunk cost. Bar-Ilan and Borodko (2019) assumes that the sunk cost a firm needs to pay in order to enter the market is a linear function the variable cost. This assumption elegantly reduces the mathematical complexity of the problem and firm's decision of entry can be explained analytically with a one dimensional state variable, namely gross mark-up. This assumption might be well-justified in some situations. However, it is restrictive and substantially limits the scope of the model's application to real world's problems. For instance, this assumption can be justified in the literature of real estate economics. Consider an individual who has the option of purchasing a house with a mortgage. The down payment, which is a fraction of the house market value, is the initial fixed cost and the mortgage monthly payments can be thought of as variable cost.

One of the main contribution of this research is relaxing this assumption and study firms' entry decision in a more general setting which is more suitable for empirical studies . Relaxing this assumption leads to a two dimensional state space problem, which requires solving a two dimensional partial differential equation with an unknown boundary for the entry decision. Problems of this kind are notoriously hard to solve and most of them have no analytical solution. Therefore, this research utilizes a numerical method based on Brekke and Øksendal (1994) and the algorithm implemented by Fackler (2018).

Our theoretical and numerical results indicate that unlike the case of Bar-Ilan and Borodko (2019), the firm's entry decision rule cannot be characterized by the firm's mark-up. Therefore, the entry decision rule in the space of variable cost and output price is not a straight line and can be concave or convex (or both in different regions) depending on the parameters. This result is reflection of the structure of the overall cost. The overall cost of operating is not a homogeneous degree one function, therefore the problem cannot be explained by the mark-up.

The most novel result of this research is that the correlation between variable cost and output price plays a crucial rule in the firm's entry decision. Most importantly, when the correlation is high, the impact of uncertainty in output price on entry depends on the level of variable cost. In the presence of high levels of correlation, increasing uncertainty in output prices increases the entry output price for low levels of variable cost and the reverse happens for high levels of variable cost. This result is in contrast with the conventional result that uncertainty increases the entry output price.

Our results indicate that by holding the rest of the parameters fixed, the firm enters at a lower output price as the correlation increases, which confirms Bar-Ilan and Borodko (2019) findings. Finally, increasing the sunk cost delays entry regardless of the level of correlation and uncertainty.

This problem can be generalized to answer a wide range questions in economics. For instance, consider a worker that has the option to accept a job offer were wage has an stochastic element, such as commission and bonuses. However, there is a sunk cost associated with accepting the job, such as relocation cost. Another example would be an employer seeking to hire a new employee, knowing that the employees efficiency is stochastic, in order to hire a new employee the employer needs to pay the sunk cost of searching for the most suitable employee and probably signing bonus upon hiring. Therefore, the solution to this problem can be used to address questions that often rises in labor economics and contract design literature.[4]

This paper is organized as follows: The model is introduced in Section 2. Section 3 establishes that the value function of an idle firm does not have a closed form solution for some rich families of function. It also addresses under what condition(s) this problem can be solved analytically. Section 4 provides a brief introduction to the numerical methods used to solve this problem. The numerical results are provided and discussed in Section 5. Finally, Section 6 concludes and outlines possible future works. The proofs of the propositions and lemmas can be found in the Appendix.

2. The Model

A risk-neutral firm has access to a production technology with an option (right) of entering the market at anytime. Upon entry the firm must incur a lump-sum sunk cost k. Let the output flow of a project be a unit of production. When the firm is active it can sell each unit of output for price p and pays a variable (marginal) cost w for its production. Once the firm is active, exit is not an option. However, this assumption can be relaxed by adding an exit lump-sum cost (or scrap value).[5] We assume that the firm is a price-taker in both input and output markets. The variable cost and output price are characterized by

[4]It is worth mentioning that there is a close connection between this problem and option theory in financial economics. A *call option* is a contract that gives the buyer the option to buy (a stock or commodity) for an upfront price (or premium). The seller is obliged to sell at a certain time or anytime depending on the type of the option, see Hull (2003) for an introduction to financial options. In this problem the sunk cost can be thought of the premium and the production line as a combination of buying a call option with an underlying asset that pays p and simultaneously issuing another option call with underlying asset that pays w.

[5]Note that adding an exit cost does not change the qualitative features of the firm's entry behavior.

two correlated geometric Brownian motion. Specifically:

$$\begin{pmatrix} dw \\ dp \end{pmatrix} = \begin{pmatrix} w\mu_w \\ p\mu_p \end{pmatrix} dt + \begin{pmatrix} w\sigma_w\rho & w\sigma_w\sqrt{1-\rho^2} \\ p\sigma_p & 0 \end{pmatrix} \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix}, \tag{2.1}$$

where dw and dp are the incremental changes in the variable cost and output price. z_1 and z_2 are independent standard Brownian motions satisfying

$$\mathbb{E}\left[dz_1\right] = \mathbb{E}\left[dz_2\right] = 0, \tag{2.2a}$$

$$\mathbb{E}\left[dz_1^2\right] = \mathbb{E}\left[dz_2^2\right] = dt \tag{2.2b}$$

$$\mathbb{E}\left[dz_1 dz_2\right] = 0. \tag{2.2c}$$

Moreover, ρ is the correlation of dp and dw. The firm discount future revenues and costs with rate a r. Moreover, $r > \mu_w$ and $r > \mu_p$. Note that this model is an extension of Dixit (1989) where the assumption that w is constant and deterministic is relaxed.

The firm's problem has two continuous state variables, p, w and one discrete variable $E \in \{0, 1\}$. When $E = 0$ the firm is idle and when $E = 1$ the firm is active. In general the firm can switch between $E = 0$ and $E = 1$ by paying the entry and exit costs. However, assuming exit is not option simplifies the problem by prohibiting transition from $E = 1$ to $E = 0$. The value of an active firm ($E = 1$) that is operating at output price p and variable price w is simply the expected discounted value of its profit $p - w$.

Proposition 2.1. *The value function of an active firm can be written as:*

$$F(p, w) = \frac{p}{r - \mu_p} - \frac{w}{r - \mu_w}. \tag{2.3}$$

Proof: See Appendix 7.1.

However, the value function of an idle firm that is waiting to enter the market, denoted by $V(p, w)$, is not as trivial as the value function of an active firm.

Proposition 2.2. *The value function of an idle firm solves*

$$\frac{1}{2}\sigma_p^2 p^2 \frac{\partial^2 V}{\partial p^2} + \sigma_p \sigma_w \rho p w \frac{\partial^2 V}{\partial p \partial w} + \frac{1}{2}\sigma_w^2 w^2 \frac{\partial^2 V}{\partial w^2} + \mu_p p \frac{\partial V}{\partial p} + \mu_w w \frac{\partial V}{\partial w} - rV = 0 \tag{2.4}$$

Proof: See Appendix 7.2.
The economic intuition behind this proof relies on the fact that an idle firm has zero payoff, therefore the capital gain of holding the option of entry, $\frac{\mathbb{E}\left[dV(p,w)\right]}{dt}$, must equals the discounted return $rV(p, w)$.

Equations 2.3 and 2.4 are not enough to uniquely solve the entry decision of the firm. Since there are two continuous state variables that the firm has to take into account, the firm's decision depends on both p and w. When p is relatively higher than w the firm is active and idle otherwise. Therefore, there are two regions in the $(p - w)$ space, one in which the firm is active and one in which the firm is idle. Moreover, the firm is indifferent at the boundary of these regions, denoted by ∂C. Therefore, the indifference condition can be written as:

$$V(p,w)|_{\partial C} = F(p,w)|_{\partial C} - k. \tag{2.5}$$

This condition is also referred to as *value matching*. Since $F(p,w)$ is strictly increasing in p and strictly decreasing in w the boundary can be expressed as a function, denoted by $\bar{p}(w)$. Therefore, the idle firm entry decision rule can be stated as:

$$E(p,w) = \begin{cases} 0 & p \leq \bar{p}(w) \\ 1 & p > \bar{p}(w) \end{cases}. \tag{2.6}$$

Intuitively $\bar{p}(w)$ is a strictly increasing function. If the price of input w increases, it is optimal for an idle firm to enter the market at a higher output price. Another set of conditions that is required to establish a unique solution of the firm's decision problem are what is referred to as *smooth-pasting* or *high-order contact*. Smooth-pasting conditions require that the directional derivatives of value functions at the points of indifference must be equal, specifically

$$\frac{\partial V}{\partial p}\Big|_{\partial C} = \frac{\partial F}{\partial p}\Big|_{\partial C}, \tag{2.7a}$$

and

$$\frac{\partial V}{\partial w}\Big|_{\partial C} = \frac{\partial F}{\partial w}\Big|_{\partial C}. \tag{2.7b}$$

The smooth pasting conditions imply that the marginal value of waiting and operating must be equal at the optimum, see Sødal (1998) . These conditions can also be understood under some no-arbitrage condition, see Dixit and Pindyck (1994):130-32. A detailed and comprehensive explanation of smooth-pasting conditions can be found in Dixit (2013). Strulovici and Szydlowski (2015) provides a rigorous mathematical treatment of smoothness of the value function and its relation to smooth-pasting for one-dimensional Itô processes.

A set of conditions is also required to ensure that there will be no "explosive" solutions, specifically:

$$\lim_{p \to 0} V(p,w) = 0, \tag{2.8a}$$

(2.8b) $$\lim_{w\to\infty} V(p,w) = 0.$$

These conditions state that there is no value in holding the entry option if the output price is zero, and similarly if variable cost is extremely high.

Definition 2.1. **(Entry Problem)** : The solution to the firm's entry problem is a set of functions $\{V(p,w), F(P,w), \bar{p}(w)\}$ that satisfies equations 2.3,2.4,2.5, 2.7, and 2.8.

3. Theoretical Investigations

Problem 2.1 is sometimes referred to as *free boundary problem*. The difference between this problem and the partial differential equations (PDE) that shows up in natural sciences and engineering is the existence of the endogenous unknown boundary $\bar{p}(w)$. One dimensional version of this problem, with underlying geometric Brownian motion has a closed form solution. However, the existence of analytical solution for this problem is not established to the author's knowledge. It is not uncommon to see manuscripts in the literature that provide an analytical solution for problems similar to Problem 2.1, see Pindyck (2000), and Pindyck (2002), see also Balikcioglu, Fackler, and Pindyck (2011) for the correction.

This section establishes that the value function of the idle firm is neither log-separable nor homogeneous of degree one. Moreover, it provides a condition (a modification to the problem set-up) under which the problem has an analytical solution.

Proposition 3.1. *Problem 2.1 has no solution of the form :*

$$V(p,w) = A\,\Pi(p)\Omega(w),$$

where Π, Ω are twice continuously differentiable functions and A is a non-zero constant.

Proof: See Appendix 7.3.

It might seem reasonable to assume that Problem 2.1 has a solution that is homogeneous of degree one. This assumption is true and can be proved for the case of stochastic sunk cost outiliend in Dixit and Pindyck (1994):207-211. However, Problem 2.1 does not accept a homogeneous of degree one solution.

Proposition 3.2. *Problem 2.1 does not have a solution of homogeneous of degree one.*

Proof: See Appendix 7.4.

This proposition establishes that it is not simply the mark-up ($m \equiv \frac{p}{w}$) that determines the firms optimal decision for entering the market.

Proposition 3.3. *Problem 2.1 has an analytical solution if the sunk cost is a linear function of marginal cost, (i.e. $k = \kappa w$). The value function of idle function $V(p, w)$ is homogeneous of degree one, and the boundary of idle and active region is a upward-sloping line crossing the origin. Moreover, the firm's entry decision is determined by mark-up.* [6]

Proof: See Appendix 7.5.

Propositions 3.1 and 3.2 indicate that problem 2.1 might not have an analytical solution. However, some quantitative features of the idle-active boundary can be established.

Proposition 3.4. *For $w = 0$, the boundary of idle and active region has the following value:*

$$\bar{p}(0) = \frac{\gamma}{\gamma - 1} k(r - \mu_p),$$

where $\bar{p}(0) \equiv \bar{p}(w)\big|_{w=0}$, and γ is the positive root of the following quadratic equation:

$$Q(x) = \frac{1}{2}\sigma_p^2 x(x-1) + \mu_p x - r.$$

Proof: See Appendix 7.6.
This result provides a partial test for the numerical results, by comparing the numerical value of the boundary at $w = 0$ with the theoretical value provided in this proposition.

4. Numerical Method

Problem 2.1 is an example of an *optimal switching models*. An introduction to optimal switching in the literature of real options can be found in Schwartz and Trigeorgis (2004). Brekke and Øksendal (1994) establishes that optimal switching models can be reduced to functional complementarity problem.

The problem of an idle firm can be written as:

$$0 = \min\{rV(p, w) - \mathcal{L}V(p, w), V(p, w) - F(p, w) + k\}, \tag{4.1}$$

where $\mathcal{L}$ is a linear differential operator[7]

$$\mathcal{L} \equiv \frac{1}{2}\sigma_p^2 p^2 \frac{\partial^2}{\partial p^2} + \sigma_p \sigma_w \rho p w \frac{\partial^2}{\partial p \partial w} + \frac{1}{2}\sigma_w^2 w^2 \frac{\partial^2}{\partial w^2} + \mu_p p \frac{\partial}{\partial p} + \mu_w w \frac{\partial}{\partial w}.$$

Brekke and Øksendal (1994) shows that equations 4.1 in addition to equations 2.8, which ensure no "explosive" solution, solves Problem 2.1. The economic intuition behind these

[6]This proposition is based on the results of Bar-Ilan and Borodko (2019).
[7]$\mathcal{L}$ is a linear differential in a sense that

(1) for every two C^2 functions f, and g, $\mathcal{L}(f + g) = \mathcal{L}f + \mathcal{L}g$
(2) for every constant $a \in \mathbb{R}$, and C^2 function f, $\mathcal{L}(af) = a\mathcal{L}f$.

conditions is that when the firm is idle it must be that $V(p,w) \geq F(p,w) - k$ and the rate of return $rV(p,w)$ must be at least as big as the capital gain of having the entry option $rV(p,w) \geq \mathcal{L}V(p,w)$.

One way of finding a numerical solution to the complementarity problem described by 4.1 is by using *projection* methods (see Judd (1996), Miranda and Fackler (2004)). In this case projection methods reduce infinite-dimensional functional problems to finite-dimensional problems on a grid in the state space. Consider a grid in the $(p-w)$ space, $G \equiv (0, p_1, ..., p_{n_p}) \times (0, w_1, ..., w_{n_w})$, one can write an approximation for $V(p,w)$ as follows:

$$\hat{V}(p,w) = \sum_{i=1}^{n_p} \sum_{j=1}^{n_w} c_{i,j} \phi(p_i, w_j),$$

where ϕ is a family of known basis functions such as Hermite polynomials, Chebyshev polynomials, piecewise linear functions, or cubic splines, see Li and Racine (2007) for a comprehensive introduction to basis functions approximation. $\{c_{ij}\}$ are constants that to be determined by *collocation methods*, which solves the optimality conditions for every point on the grid G. Since $F(p,w)$ has an analytical form, substituting $\hat{V}(p,w)$ and $F(p,w)$ in 4.1 yields an complementarity problem which is referred to as *vertical linear complementarity problem* which is a generalization of standard linear complementarity problem.

The most crucial element of Problem 2.1 is the boundary where the idle firm chooses to become active, the boundary (approximately) consists of the points on the grid such that $\hat{V}(p,w) - F(p,w) + k$ is less than some tolerance level ϵ. A detailed and comprehensive treatment and implementation (in MATLAB) of this algorithm for optimal switching models can be found in Fackler (2018).

There are two ways to implement the numerical solution for this specific case. First, by allowing the firm to exit by paying an exit cost f and choose an arbitrary high exit cost. Second, by directly solving the problem and not allowing exit at all. The ultimate goal of this research is to empirically estimate these parameters. In our data set the firms exit the market at a positive price. The exit decision is informative and can be used to estimate the exit parameter f. Therefore, in this research the first approach is chosen. It is worth mentioning that the nature of this problem restricts the choice of family of basis functions. It is known that optimal switching problems are not very smooth at the boundary. Smooth-pasting conditions establish that the first order derivatives are continuous, but the second order derivatives are usually not. For this reason piecewise linear functions are chosen to capture second-order discontinuity at the boundary.

5. Numerical Results

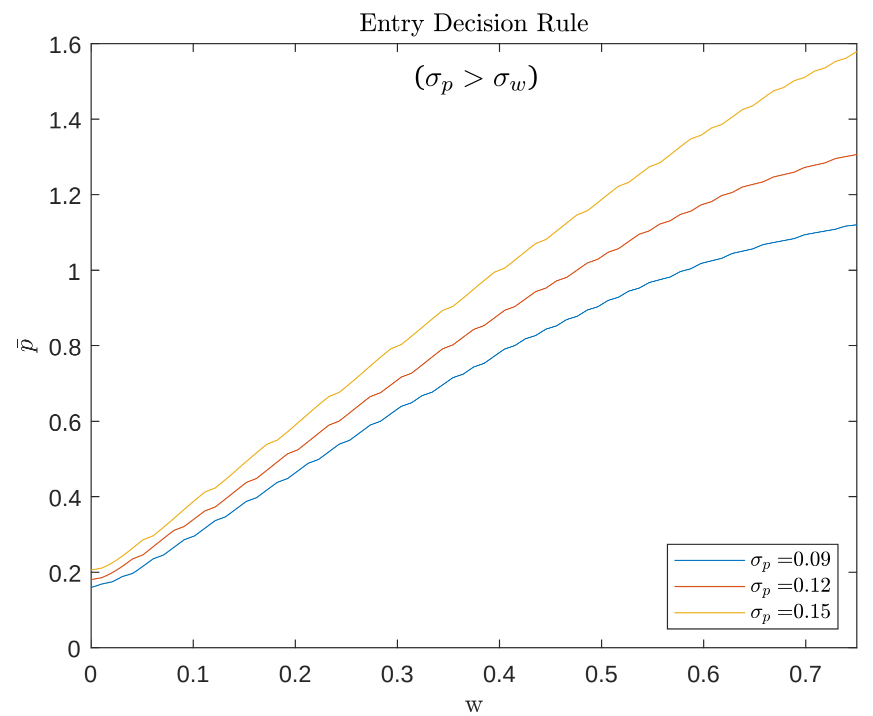
This section provides the numerical solution to Problem 2.1. The most crucial feature of this model is the boundary between a idle and active firm, i.e $\bar{p}(w)$. Figures 1-5 provide comparative statics for $\bar{p}(w)$ for different sets of parameters. Since the aim of this research is to study the effect of uncertainty in output price (σ_p) and the uncertainty in variable cost (σ_w) in the presence of correlation (ρ) between these two processes, the drift parameters (μ_p, μ_w) are held constant at zero.

Figure 1 confirms the first theoretical contribution of this paper established in Proposition 3.2, i.e., it is not simply the mark-up that determines the firm's optimal decision for entering the market. As shown by Bar-Ilan and Borodko (2019), when mark-up is the only determinant of entry decision then $\bar{p}(w)$ should be a linear function. However as shown in Figures 1-5, $\bar{p}(w)$ is generally a nonlinear function. This result is a reflection of the cost structure of the entry problem. The expected present value of the firm's overall cost is $k - \frac{w}{r-\mu_w}$, which is not a homogeneous of degree one function. Therefore, the value function of an idle firm cannot be homogeneous of degree one and as a consequence the entry decision problem cannot be determined by the firm's mark-up.

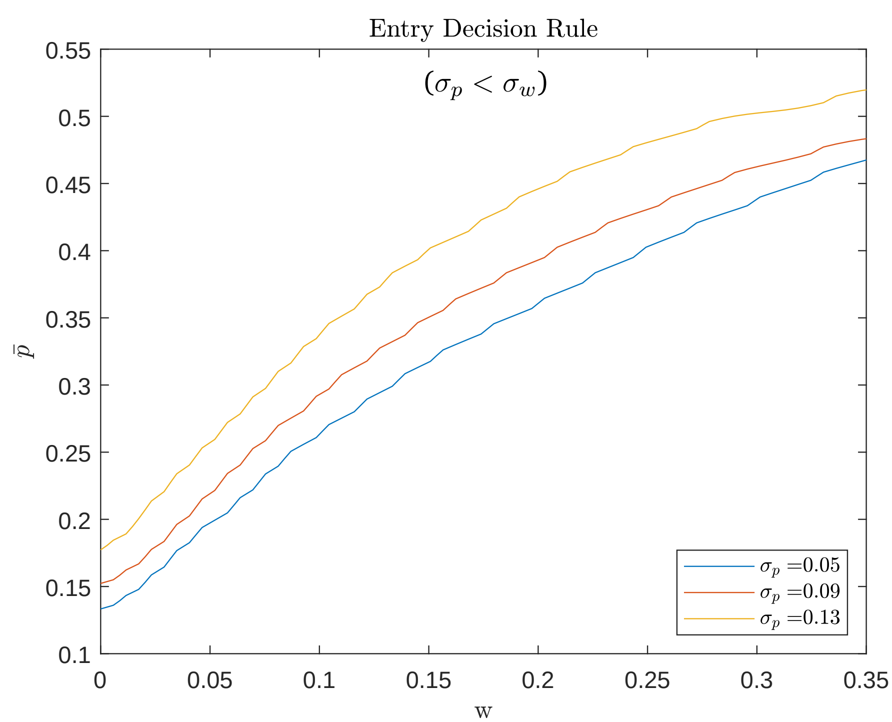
Figure 1 also illustrates the impact of uncertainty in output prices on $\bar{p}(w)$ when the correlation is zero for two different case : (a) $\sigma_p > \sigma_w$ and, (b) $\sigma_p < \sigma_w$. In both cases increasing σ_p creates an upward shift in the boundary. Therefore, regardless of the values of σ_w and σ_p, increasing σ_p increases the entry output price. Since dp and dw are independent and σ_w is held fixed, similar to the one-dimensional case (i.e. Dixit (1989)), increasing the uncertainty in output price increases the value function of the idle firm. Therefore, it is optimal for the firm to wait longer in order to obtain more information about the price.

Figure 2 illustrates the main findings of this paper. These figures show the impact of uncertainty in output prices on $\bar{p}(w)$ when the correlation is very high for two different case : (a) σ_p is sufficiently larger than σ_w and, (b) σ_p ranges from $\sigma_p < \sigma_w$ to sufficiently larger than σ_w.

As shown in Figure 2a, when the uncertainty in output price is sufficiently higher than the uncertainty in variable cost, increasing σ_p increases the output price (for a given w). This results can be explained by considering two limiting case. When w is small (i.e. $\frac{w}{r-\mu_w} \ll k$) the overall cost of entry is dominated by the fixed cost and the marginal cost is negligible. This case is (approximately) the same as the one-dimensional problem provided by Dixit (1989). In this case increasing σ_p increases the output price that the idle firm enters the market. When w is large (i.e. $\frac{w}{r-\mu_w} \gg k$), the overall cost of entry is



(A) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \rho = 0, \sigma_w = 0.08, r = 0.025, k = 4.0)$



(B) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \rho = 0, \sigma_w = 0.15, r = 0.025, k = 4.0)$

FIGURE 1. The impact of the output price uncertainty (σ_p) on the entry decision rule, with zero correlation between input and output price.

dominated by the variable cost and the fixed cost is negligible. In this case, as shown in Proposition 3.3, the problem can be reduced to a one-dimensional problem where the mark-up determines the optimal entry rule. When σ_p is sufficiently larger than σ_w, the mark-up volatility is a strictly increasing function of σ_p. Therefore, increasing σ_p creates an upward shift in $\bar{p}(w)$. Since increasing σ_p creates an upward shift in both limiting cases and $\bar{p}(w)$ is a strictly increasing function of w, increasing σ_p creates an upward shift in the boudary for all values of w.

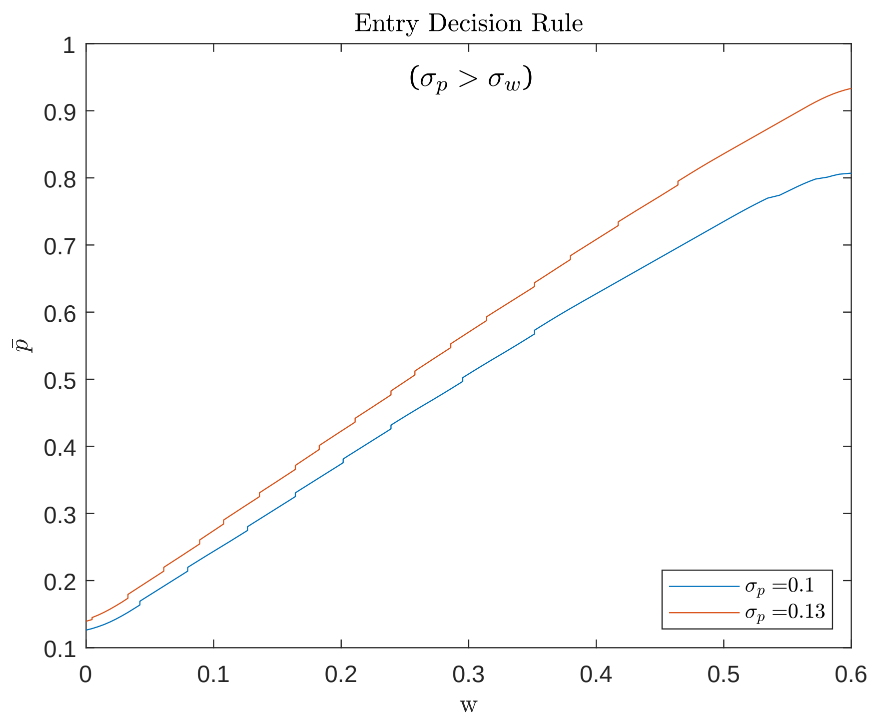
However, as shown in Figure 2b when σ_p moves from 0.05 $(\sigma_p < \sigma_w)$ to 0.1 $(\sigma_p > \sigma_w)$ the impact of uncertainty reverses depending on the level of the variable cost. In this case there is a crossing in the boundaries of idle-active regions. When w is small, increasing σ_p increases the output entry price. When w is large, increasing σ_p decreases the output entry price. This is a novel result, that to the best of the author's knowledge, has never been discussed in the literature. One way to understand this result is by considering those two limiting cases. When w is very small the overall cost of entering is dominated by the sunk cost k and the variable cost is negligible. Therefore, the results are very similar to the one-dimensional case where only the output price is stochastic. Therefore, increasing σ_p increases the entry output price.

When w is very large, the sunk cost becomes negligible. And the predominant cost of entry is the expected discounted present value of variable cost. In this case again as shown in Proposition 3.3, the firm's entry decision depends on the mark-up. The effect of uncertainty in output price on the uncertainty of mark-up can be written as:

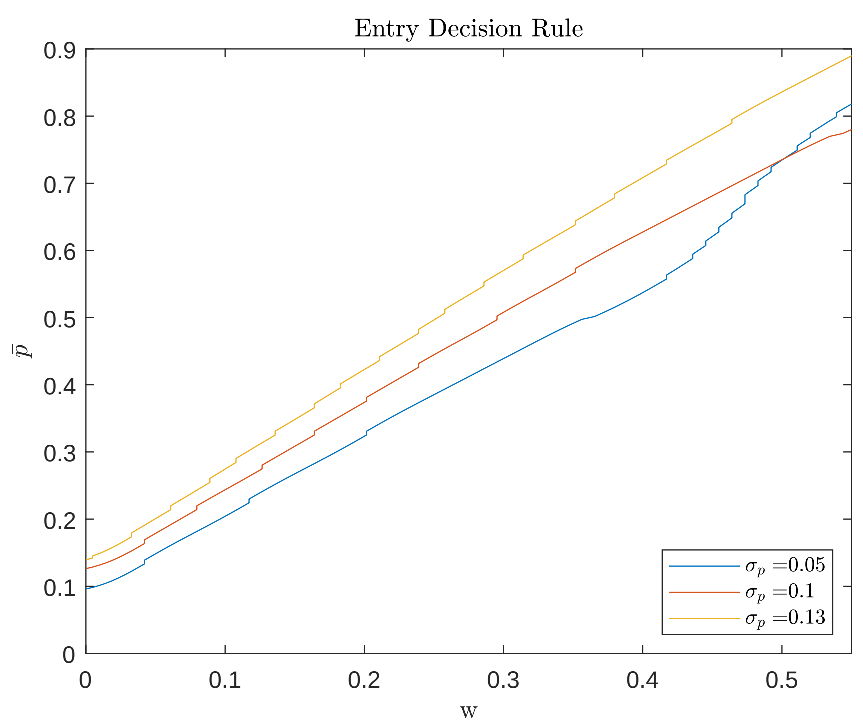
$$\frac{\partial \sigma_m^2}{\partial \sigma_p} = \frac{\partial}{\partial \sigma_p}\left[\sigma_p^2 - 2\sigma_p\sigma_w\rho + \sigma_w^2\right] = 2\big(\sigma_p - \rho\sigma_w\big).$$

When $\sigma_p < \rho\sigma_w$, increasing the uncertainty in output price decreases the overall uncertainty in mark-up. Therefore, the firm enters at a lower output price (for a given w). Since the boundary is a continuous function there must be a w^* in the middle where these two boundaries cross each other.

Figure 3 illustrates the impact of the correlation between dp and dw on $\bar{p}(w)$ for two different cases: (a) $\sigma_p > \sigma_w$, and (b) $\sigma_p < \sigma_w$. In both cases increasing the correlation creates a downward shift in the boundary. Therefore, regardless of the values of σ_p, and σ_w, the firm enters the market at a lower output price (for a given w) or higher variable price (given for a given w) for higher levels of correlation. The firm's overall uncertainty stems from two sources, namely, output price and variable cost. When the marginal cost is high, the fixed cost is negligible and the entry problem can be explained by a special case of the model provided by Bar-Ilan and Borodko (2019), namely the case of zero sunk



(A) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \sigma_w = 0.08, r = 0.025, \rho = 0.7, k = 3.0)$



(B) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \sigma_w = 0.08, r = 0.025, \rho = 0.7, k = 3.0)$

FIGURE 2. The impact of the output price uncertainty (σ_p) on the entry decision rule, with positive correlation between input and output price.

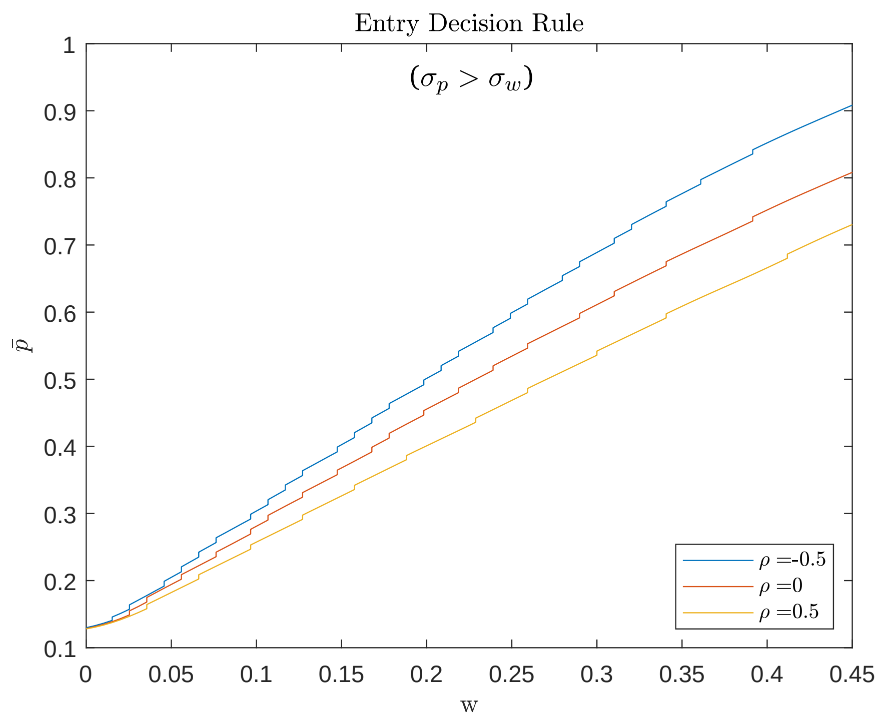
cost. In this case mark-up determines the entry decision and as shown in the proof of Proposition 3.3 the mark-up volatility can be written as:

$$\sigma_m^2 = \sigma_p^2 - 2\sigma_p\sigma_w\rho + \sigma_w^2.$$

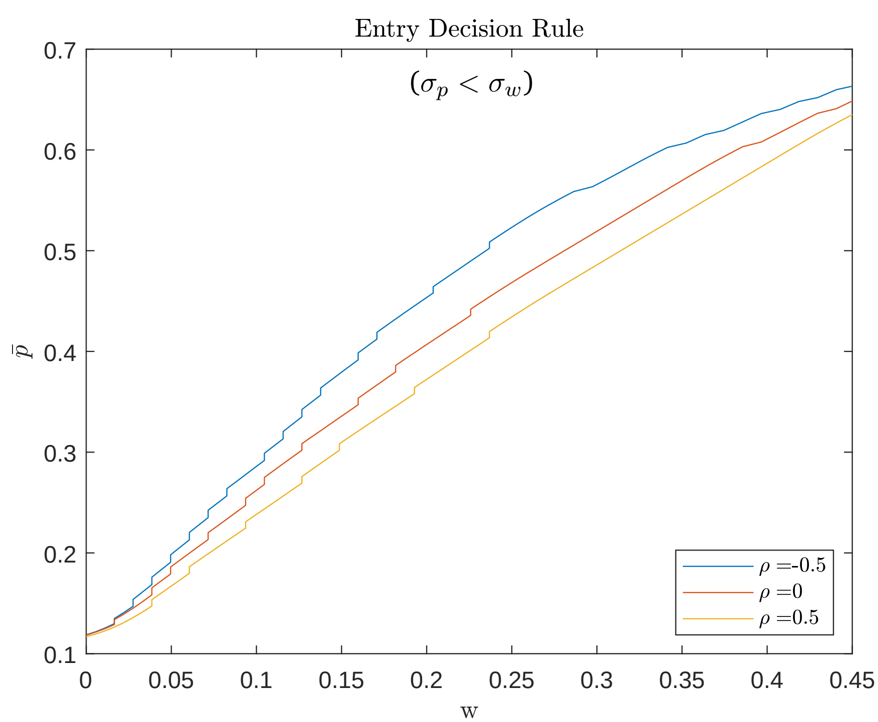
Therefore, increasing the correlation decreases the volatility of mark-up which consequently creates downward shift in the boundary. However, due to the non-linear nature of this problem and lack of an analytical solution, the same argument cannot be made for low and mid-range levels of marginal cost. A better understanding of the effect of correlation on entry price requires a thorough theoretical investigation.

Figure 4 illustrates the impact of changing the sunk cost on $\bar{p}(w)$ when the correlation between dp and dw is zero for two different cases : (a) $\sigma_p > \sigma_w$, and (b) $\sigma_p < \sigma_w$. When the correlation is zero, increasing the sunk cost creates an upward shift in the boundary.

Figure 5 illustrates the impact of changing the sunk cost on $\bar{p}(w)$ when the correlation between dp and dw is non-zero for two different cases : (a) $\rho > 0$, and (b) $\rho < 0$. When the correlation is non-zero, increasing the sunk cost creates an upward shift in the boundary. Therefore, Figures 4 and 5 illustrate that regardless of the values of σ_p, σ_w, and ρ the firm enters the market at a higher output price (for a given w) or lower variable price (given for a given p) for higher sunk cost. Intuitively, when the sunk cost increases the firm has to be compensated with higher output price (lower variable cost) to be indifferent between being active and idle.

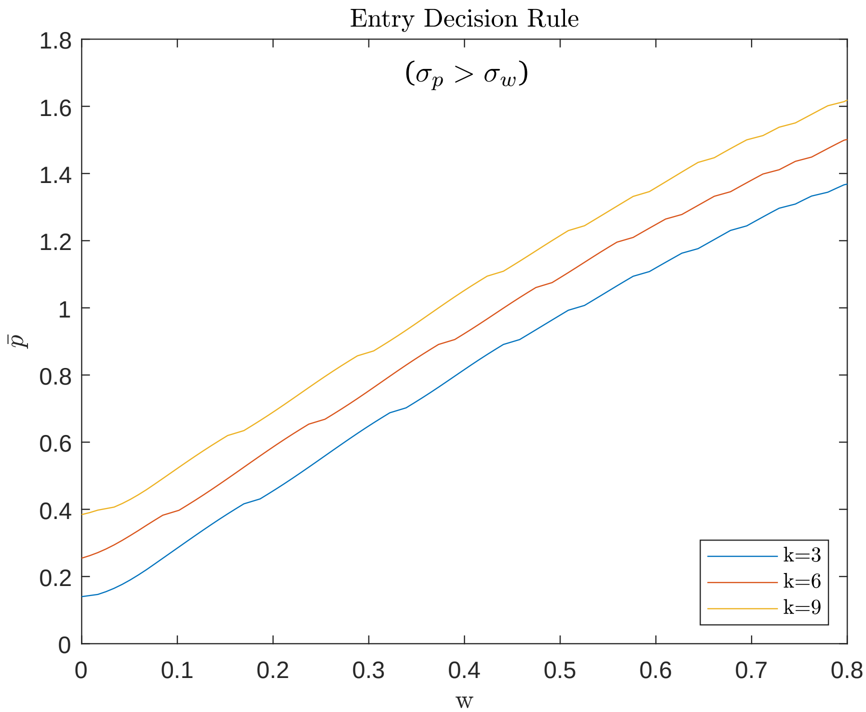


(A) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \sigma_w = 0.08, \sigma_p = 0.1, r = 0.025, k = 3.0)$

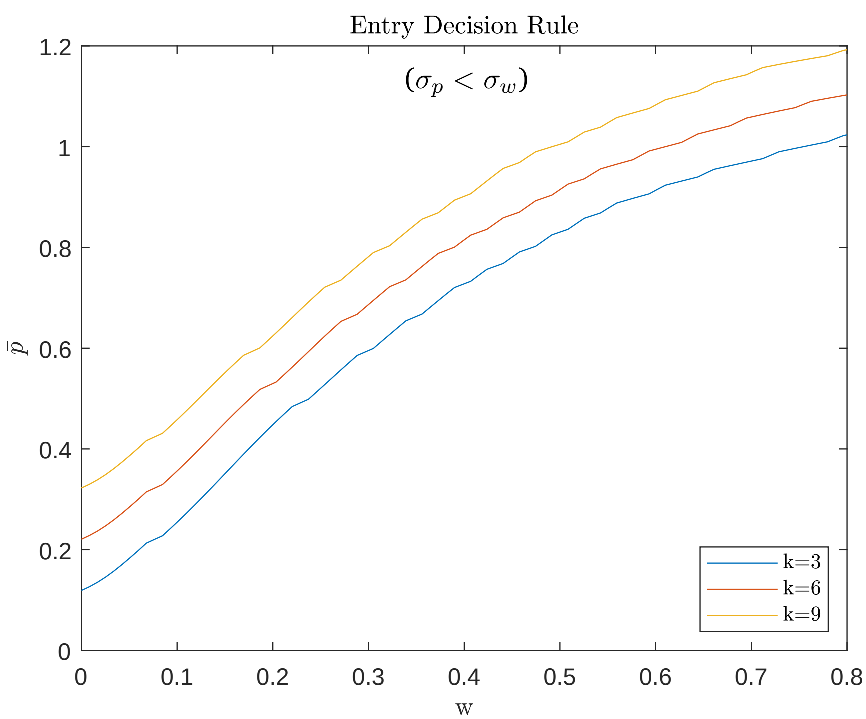


(B) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \sigma_w = 0.1, \sigma_p = 0.08, r = 0.025, k = 3.0)$

FIGURE 3. The impact of correlation between variable cost and output price on the entry decision rule.

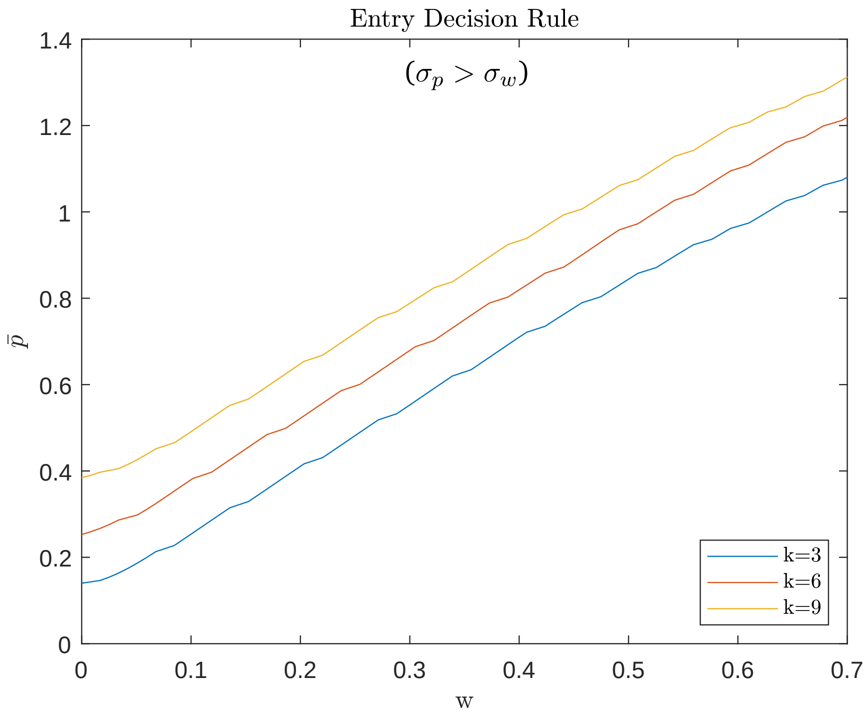


(A) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \rho = 0, \sigma_w = 0.08, \sigma_p = 0.11, r = 0.025)$

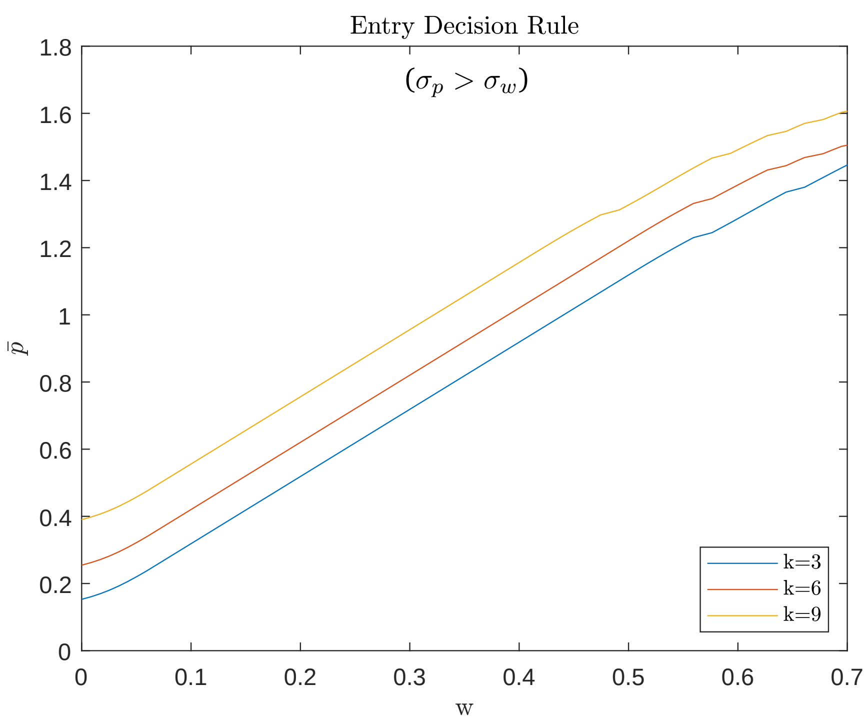


(B) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \rho = 0, \sigma_w = 0.11, \sigma_p = 0.08, r = 0.025)$

FIGURE 4. The impact of sunk cost (k) on the entry decision rule, with zero correlation between output and input price.



(A) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \rho = 0.5, \sigma_w = 0.08, \sigma_p = 0.11, r = 0.025)$



(B) The boundary $\bar{p}(w)$, for paramters $(\mu_p = \mu_w = 0, \rho = -0.5, \sigma_w = 0.08, \sigma_p = 0.11, r = 0.025)$

FIGURE 5. The impact of sunk cost (k) on the entry decision rule, with non-zero correlation between output and input price.

6. Conclusion

Most of the models of investment with uncertainty only consider one source of uncertainty, for instance uncertainty in output prices or input prices. This assumption significantly reduces the mathematical complexity of these problems. However, it severely restricts the applicability of these models to real world problems of investment. This paper provide a more realistic model that the firm faces two sources of uncertainty, namely output price and variable cost.

This paper studies the problem of a price-taker and risk neutral firm has access to a production technology with an option (right) of entering the market at anytime. Upon entry the firm must incur a lump-sum sunk cost k. The output price and variable cost are characterized by two correlated geometric Brownian motions. The theoretical results highly indicate that this problem does not have an analytical solution. Therefore, a numerical approach is utilized.

The theoretical results established that, as opposed to Bar-Ilan and Borodko (2019), mark-up cannot be the only determinant of the firm's entry decision rule. The entry decision can be characterized by a strictly increasing curve $\bar{p}(w)$ in the space of marginal cost and output price. This curve is generally nonlinear and depending on the parameters can be either concave or convex, or combination of both in different regions. This result is a reflection of the cost structure of the entry problem. The expected present value of the firm's overall cost is $k - \frac{w}{r-\mu_w}$, which is not a homogeneous of degree one function. Therefore, the value function of an idle firm cannot be homogeneous of degree one and as a consequence the entry decision problem cannot be determined by the firm's mark-up.

The numerical results indicate that for high levels of correlation the impact of uncertainty in output prices (σ_p) on entry price is ambiguous and depends on the variable cost. Specifically, when $\sigma_p > \sigma_w$ the conventional result that increasing σ_p creates an upward shift in $\bar{p}(w)$ holds. However, when $\sigma_p < \sigma_w$ this relation depends on w, when w is small increasing σ_p increases $\bar{p}(w)$ and when w is large this relation is reversed. This novel effect is a reflection of non-linearity of this problem. This results can be explained by considering two limiting cases that have analytical solutions. When w is small, the overall cost of operating is dominated by the fixed cost and the variable cost is negligible. This case is the approximately the same as Dixit (1989) problem which entry price is an increasing function of σ_p. When w is large, the overall cost of operating is dominated by the variable cost and the sunk cost is negligible. In this case the overall cost is homogeneous of degree one and the mark-up determines the entry decision. When $\sigma_p < \sigma_w$ and correlation level is high the uncertainty in mark-up is a decreasing function of σ_p. In this regime, when w is sufficiently large the entry price is a decreasing function of σ_p. Therefore, the effect of σ_p on entry reverses as w increases.

The numerical results indicate that regardless of the parameters, increasing the correlation decreases the overall uncertainty. Therefore, the firm enters the market at a lower output price, which confirms the findings of Bar-Ilan and Borodko (2019). When the variable cost is high the entry behavior can be explained by the firm's mark-up. The mark-up uncertainty is a decreasing function of the correlation, therefore, increasing ρ decreases the uncertainty in the mark-up and creates a downward shift in the boundary of idle-active regions. However, due to the non-linear nature of this problem and lack of an analytical solution, the same argument cannot be made for low and mid-range levels of marginal cost. A better understanding of the effect of correlation on entry price requires a thorough theoretical investigation.

Finally, the theoretical (Proposition 3.4) and numerical results show that regardless of the parameters, increasing the sunk cost increases the output entry price for a given level of variable cost. Since the sunk cost is deterministic, increasing sunk cost increases the overall cost of operating and consequently creates an upward shift in the boundary of idle-active regions.

7. Appendix

7.1. Proof of Proposition 2.1

Since the active firm has no exit option, it collects the annuity flow of $p - w$, therefore the value function is the discounted net present value,

$$F(p, w) = \mathbb{E}\left[\int_0^{\infty} e^{-rt}\big(p(t) - w(t)\big)dt\right],$$

changing the order of integration and expectation one can write,

$$F(p, w) = \int_0^{\infty} e^{-rt}\big(pe^{\mu_p t} - we^{\mu_w t}\big)dt = \frac{p}{r - \mu_p} - \frac{w}{r - \mu_w}.$$

■

A more detailed proof of this proposition for one-dimensional case can be found in Stokey (2009).

7.2. Proof of Proposition 2.2

Let $(z_1, ..., z_n)$ denote an n-dimensional Brownian motion, and $(s_1, ..., s_m)$ denote an m-dimensional Itô processes as follows:

$$\begin{pmatrix} ds_1 \\ \vdots \\ ds_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nm} \end{pmatrix} \begin{pmatrix} dz_1 \\ \vdots \\ dz_m \end{pmatrix},$$

or in a matrix form

$$ds = \mu dt + \sigma dz$$

where μ_i, and $\sigma_{i,j}$ satisfies some regular conditions stated in definition 4.1.1 in Øksendal (2003) for all $i \in \{1, ..., n\}$, and $j \in \{1, ..., m\}$.

Theorem 7.1. (Multivariate Itô formula) *Let ds be an n-dimensional Itô process. Let $v(t, s)$ be a $\mathcal{C}^2$ map from $[0, \infty] \times \mathbb{R}^n$ to $\mathbb{R}^p$. Then the process $y(t, s) \equiv v(t, s)$ is another process such that $\forall k \in \{1, ..., p\}$*

$$dy_k = \frac{\partial g_k}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial s_i}ds_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g_k}{\partial s_i \partial s_j}ds_i ds_j.$$

See Øksendal (2003) for the proof.

Since the idle firm has zero payoff, its value function at time t (corresponding to (p, w)) must equals its discounted expected value in an infinitesimal change in time

$$V(p,w) = \frac{1}{1+rdt}\mathbb{E}\big[V(p,w) + dV(p,w)\big].$$

Using multivariate Itô formula for $dV(p, w)$

$$\mathbb{E}\big[dV(p,w)\big] = \mathbb{E}\left[\frac{\partial V}{\partial p}dp + \frac{\partial V}{\partial w}dw + \frac{1}{2}\frac{\partial^2 V}{\partial p^2}dp^2 + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}dw^2 + \frac{\partial^2 V}{\partial p \partial w}dpdw\right].$$

Applying the expectation operator and preserving only $\mathcal{O}(dt)$ terms, one can write

$$\mathbb{E}\big[dV(p,w)\big] = \mu_p p\frac{\partial V}{\partial p} + \mu_w w\frac{\partial V}{\partial w}w + \frac{1}{2}\sigma_p^2 p^2\frac{\partial^2 V}{\partial p^2} + \frac{1}{2}\sigma_w^2 w^2\frac{\partial^2 V}{\partial w^2} + \rho\sigma_p\sigma_w pw\frac{\partial^2 V}{\partial p \partial w}.$$

Using the first order expansion of $\frac{1}{1+rdt} = 1 - rdt$ and rearranging yield

$$\frac{1}{2}\sigma_p^2 p^2\frac{\partial^2 V}{\partial p^2} + \sigma_p\sigma_w\rho pw\frac{\partial^2 V}{\partial p \partial w} + \frac{1}{2}\sigma_w^2 w^2\frac{\partial^2 V}{\partial w^2} + \mu_p p\frac{\partial V}{\partial p} + \mu_w w\frac{\partial V}{\partial w} - rV = 0$$

■

7.3. Proof of Proposition 3.1

Suppose there exists a solution of the form

$$V(p,w) = A\Pi(p)\Omega(w).$$

Without a loss of generality assume $\rho = 0$. Substituting the ansatz in the inactive firm's Hamilton-Jacobi-Bellman yields

$$\frac{1}{2}\sigma_p^2 p^2\Pi''(p)\Omega(w) + \frac{1}{2}\sigma_w^2 w^2\Omega''(w)\Pi(p) + \mu_p p\Pi'(p)\Omega(w) + \mu_w w\Omega'(w)\Pi(p) - r\Pi(p)\Omega(w) = 0.$$

Dividing by $\Pi(p)\Omega(w)$ and rearranging yields

$$\frac{1}{2}\sigma_p^2 p^2\frac{\Pi''(p)}{\Pi(p)} + \mu_p p\frac{\Pi'(p)}{\Pi(p)} = r - \frac{1}{2}\sigma_w^2 w^2\frac{\Omega''(w)}{\Omega(w)} - \mu_w w\frac{\Omega'(w)}{\Omega(w)}.$$

Considering the left-hand side is only a function of p and the right-hand side is only a function of w, one can write

$$\frac{1}{2}\sigma_p^2 p^2\frac{\Pi''(p)}{\Pi(p)} + \mu_p p\frac{\Pi'(p)}{\Pi(p)} = c,$$

$$r - \frac{1}{2}\sigma_w^2 w^2\frac{\Omega''(w)}{\Omega(w)} - \mu_w w\frac{\Omega'(w)}{\Omega(w)} = c,$$

where c is a constant. The last two equations are ordinary differential equations with the following solutions:

$$\Pi(p) = B_p p^{\alpha};$$
$$\Omega(w) = B_w w^{\beta},$$

where B_p and B_w are some constants such that $A = B_p B_w$. The pair (α, β) must satisfy

$$\frac{1}{2}\sigma_p^2 \alpha(\alpha - 1) + \mu_p \alpha + \frac{1}{2}\sigma_w^2 \beta(\beta - 1) + \mu_w \beta - r = 0, \tag{7.3}$$

and

$$\alpha > 0, \beta < 0.$$

The last two inequalities ensure that boundary condition described by Equation 2.8 are not violated.

Lemma 7.1. *The intersection of* $\{(\alpha, \beta) : \alpha > 0, \beta < 0\}$ *and the set of* (α, β) *that satisfies Equation 7.3 is non-empty. Moreover, the positive root of Equation 7.3 for* $\beta = 0$ *, is larger than one.*

Proof: Equation 7.3 is as an ellipse (see Figure 6) with following standard form:

$$\frac{\left(\alpha - \alpha_0\right)^2}{a_0^2} + \frac{\left(\beta - \beta_0\right)^2}{b_0^2} = 1$$

where

$$\alpha_0 \equiv \frac{1}{2} - \frac{\mu_p}{\sigma_p^2},$$
$$\beta_0 \equiv \frac{1}{2} - \frac{\mu_w}{\sigma_w^2},$$
$$a_0^2 \equiv \frac{2r}{\sigma_p^2} + \alpha_0^2 + \frac{\sigma_w^2}{\sigma_p^2}\beta_0^2,$$
$$b_0^2 \equiv \frac{2r}{\sigma_w^2} + \frac{\sigma_p^2}{\sigma_w^2}\alpha_0^2 + \beta_0^2.$$

Therefore, the right vertex of the ellipse can be written as:

$$v_+^{\alpha} \equiv \alpha_0 + a_0 = \alpha_0 + \sqrt{\frac{2r}{\sigma_p^2} + \alpha_0^2 + \frac{\sigma_w^2}{\sigma_p^2}\beta_0^2} > 0$$

given $r > 0$. The bottom vertex of the ellipse can be written as:

$$v_-^{\beta} \equiv \beta_0 - b_0 = \beta_0 - \sqrt{\frac{2r}{\sigma_w^2} + \frac{\sigma_p^2}{\sigma_w^2}\alpha_0^2 + \beta_0^2.} < 0$$

given $r > 0$.

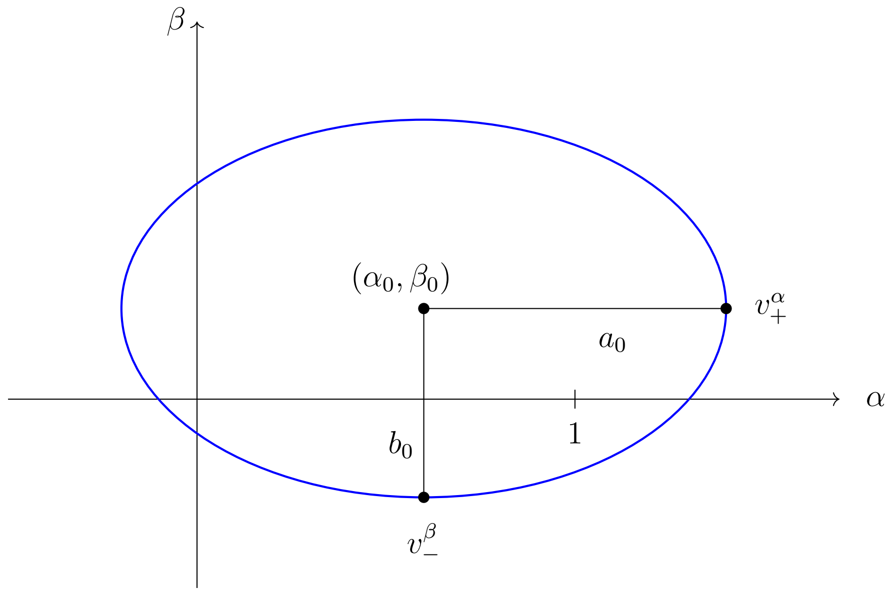


FIGURE 6. The standard form of the ellipse described by Eq. 7.3

Therefore, the set of $\{(\alpha, \beta) : \alpha > 0, \beta < 0\}$ that satisfies Equation 7.3 is not empty.

When β is zero, Equation 7.3 can be written as:

$$\frac{1}{2}\sigma_p^2\alpha(\alpha-1) + \mu_p\alpha - r = 0,$$

which has two roots as follows:

$$\alpha_\pm \equiv \frac{1}{2} - \frac{\mu_p}{\sigma_p^2} \pm \sqrt{\frac{2r}{\sigma_p^2} + \left(\frac{1}{2} - \frac{\mu_p}{\sigma_p^2}\right)^2}.$$

By assumption $r > \mu_P$, therefore, one can write

$$\alpha_+ > \frac{1}{2} - \frac{\mu_p}{\sigma_p^2} + \sqrt{\frac{2\mu_p}{\sigma_p^2} + \left(\frac{1}{2} - \frac{\mu_p}{\sigma_p^2}\right)^2} = \frac{1}{2} - \frac{\mu_p}{\sigma_p^2} + \sqrt{\left(\frac{1}{2} + \frac{\mu_P}{\sigma_P^2}\right)^2} = 1.$$

■

The value matching and smooth pasting conditions can be written as:

$$A\bar{p}(w)^\alpha w^\beta = \frac{\bar{p}(w)}{r-\mu_p} - \frac{w}{r-\mu_w} - k, \tag{7.6}$$

$$A\alpha\bar{p}(w)^{\alpha-1}w^\beta = \frac{1}{r-\mu_p}, \tag{7.7a}$$

$$A\beta\bar{p}(w)^\alpha w^{\beta-1} = -\frac{1}{r-\mu_w}. \tag{7.7b}$$

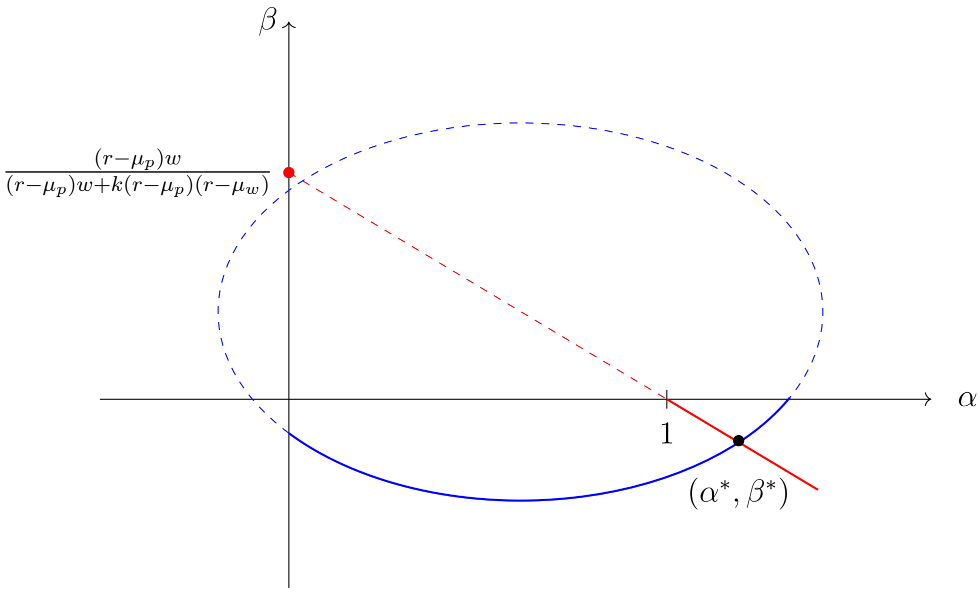


FIGURE 7. Existence and uniqueness of (α^*, β^*).

Dividing Equation 7.7a by Equation 7.7b and rearranging yields

$$\bar{p}(w) = -\left(\frac{\alpha}{\beta}\right)\frac{r-\mu_p}{r-\mu_w}w. \tag{7.8}$$

Similarly, dividing Equation 7.6 by Equation 7.7a and rearranging yields

$$\bar{p}(w) = \frac{\alpha}{\alpha-1}\left[\frac{r-\mu_p}{r-\mu_w}w + k(r-\mu_p)\right]. \tag{7.9}$$

Equating the left-hand side of equations 7.8 and rearranging 7.9 yields

$$\left[\frac{r-\mu_p}{r-\mu_w}w + k(r-\mu_p)\right]\beta + \left(\frac{r-\mu_p}{r-\mu_w}w\right)\alpha = \frac{r-\mu_p}{r-\mu_w}w. \tag{7.10}$$

Lemma 7.2. *There exist a unique (α^*, β^*) which satisfies $\alpha^* > 0$, $\beta^* < 0$, Equation 7.10, and Equation 7.3. Moreover, $\alpha^* > 1$.*

Proof:

Note Equation 7.10 describes a downward sloping line in the space of (α, β). The β-intercept can be written as $\frac{(r-\mu_p)w}{(r-\mu_p)w+k(r-\mu_p)(r-\mu_w)}$ which is between zero and one, and the α-intercept is 1 (see Figure 7). As shown in Lemma 7.1, the intersection of the ellipse and $\{(\alpha, \beta) : \alpha > 0, \beta < 0\}$ is non-empty. Moreover, when β is zero the larger α on the ellipse is larger than one. Therefore, (α^*, β^*) is unique and $\alpha^* > 1$. ■

Therefore, if a solution of the form $A\Pi(p)\Omega(w)$ exists, it should be in the form of $Ap^\alpha w^\beta$, and for every w, (α, β) uniquely solves equations 7.3, and 7.10. The boundary where the firm is indifferent between entry and waiting is uniquely determined with equation 7.9 (or

7.8). The constant A can be found by substituting $\bar{p}(w)$, α, and β in the value matching condition (equation 7.6) or one the smooth pasting conditions (equations 7.7a, 7.7b).

Now consider the following parameters ($\sigma_p = 0.1, \sigma_w = 0.1, \mu_p = 0.0, \mu_w = 0.0, r = 0.05, k = 5$). Setting $w = 1$ and solving the model for A yields

$$A\Big|_{w=1} = 1.92,$$

and setting $w = 2$ and solving the model for A yields

$$A\Big|_{w=2} = 2.14.$$

This result contradicts the assumption that A is constant and not a function of w. ■

7.4. Proof of Proposition 3.2

Assume that $V(p, w)$ is homogeneous of degree one. Assuming $w \neq 0$, one can write

$$V(p, w) = wV\left(\frac{p}{w}, 1\right) \equiv wv(m),$$

where $m \equiv \frac{p}{w}$, m can be interpreted as the gross mark-up of the firm. The HJB of the idle firm can be written as:

$$\frac{1}{2}\left(\sigma_p^2 - 2\sigma_p\sigma_w\rho + \sigma_w^2\right)m^2 v''(m) + \left(\mu_p - \mu_w\right)mv'(m) - \left(r - \mu_w\right)v(m) = 0.$$

The value matching condition can be written as:

$$v(\bar{m}) = \frac{\bar{m}}{r - \mu_p} - \frac{1}{r - \mu_w} - \frac{k}{w},$$

and the smooth-pasting conditions can be written as:

$$v'(\bar{m}) = \frac{1}{r - \mu_p},$$

$$v(\bar{m}) - mv'(\bar{m}) = -\frac{1}{r - \mu_w}.$$

Combining the smooth-pasting conditions and rearranging yields

$$v(\bar{m}) = \frac{\bar{m}}{r - \mu_p} - \frac{1}{r - \mu_w},$$

which contradicts the value matching condition for any non-zero and finite values of k, and w. ■

7.5. Proof of Proposition 3.3

Defining m as the gross mark-up, the HJB equation of an idle firm can be written as:

$$\frac{1}{2}\big(\sigma_p^2 - 2\sigma_p\sigma_w\rho + \sigma_w^2\big)m^2 v''(m) + \big(\mu_p - \mu_w\big)mv'(m) - \big(r - \mu_w\big)v(m) = 0. \tag{7.12}$$

The value matching condition can be written as:

$$v(\bar{m}) = \frac{\bar{m}}{r - \mu_p} - \frac{1}{r - \mu_w} - \kappa,$$

and the smooth-pasting conditions can be written as:

$$v'(\bar{m}) = \frac{1}{r - \mu_p},$$

$$v(\bar{m}) - mv'(\bar{m}) = -\frac{1}{r - \mu_w} - \kappa.$$

Note that one of the smooth-pasting conditions is redundant. Therefore, in order to solve the problem of the idle firm the value matching and the first smooth-pasting condition is sufficient. Note that HJB equation of the idle firm is a linear ordinary differential equation. Therefore, using $v(m) = Cm^\eta$ as an ansatz, the HJB equation can be written as:

$$\frac{1}{2}\big(\sigma_p^2 - 2\sigma_p\sigma_w\rho + \sigma_w^2\big)\eta(\eta - 1) + \big(\mu_p - \mu_w\big)\eta - \big(r - \mu_w\big) = 0. \tag{7.14}$$

One can show that Equation 7.14 has a negative root and a positive root. Moreover, the positive root (η_+) is larger than one. The value function of an idle firm can be written as:

$$V(p, w) = Cp^{\eta_+}w^{1-\eta_+},$$

which is homogeneous of degree one. And $\bar{p}(w)$ can be written as:

$$\bar{p}(w) = \left(\frac{\eta_+}{\eta_+ - 1}\right)\left(\frac{r - \mu_p}{r - \mu_w}\right)\big[1 + (r - \mu_w)\kappa\big]w.$$

■

For a comprehensive and detailed treatment of this problem see Bar-Ilan and Borodko (2019).

7.6. Proof of Proposition 3.4

Since w is a geometric Brownian motion, $w = 0$ is an absorbing barrier. If $w = 0$ is observed then the idle firm expects w stays at zero indefinitely. Therefore, the problem is reduced to a one-dimensional one as follows:

$$\frac{1}{2}\sigma_p^2 p^2 \frac{d^2V}{dp^2} + \mu_p p \frac{dV}{dp} - rV = 0,$$

$$V(\bar{p}(0)) = \frac{\bar{p}(0)}{r - \mu_p} - k,$$
$$\frac{dV}{dp}\Big|_{p=\bar{p}(0)} = \frac{1}{r - \mu_p}.$$
$$\lim_{p \to 0} V(p) = 0.$$

Substituting $V = Dp^x$ as an ansatz yields

$$Q(x) \equiv \frac{1}{2}\sigma_p^2 x(x-1) + \mu_p x - r = 0.$$

It can be shown this quadratic equation has a positive and a negative root. Moreover, the positive root, denoted by γ is larger than one. Rejecting the explosive root (the negative root), and substituting the ansatz in the value-matching and smooth-pasting equations and rearranging yields

$$\bar{p}(0) = \frac{\gamma}{\gamma - 1} k(r - \mu_p).$$

■

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